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ABSTRACT

This course is intended for students who have a thorough knowledge of college preparatory mathematics including algebra, axiomatic geometry, trigonometry, and analytic geometry. It does not assume they have acquired a background of elementary functions. This teacher's guide contains background information, suggested instructional procedures, and answers to student exercises. (RH)

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SCHOOL
MATHEMATICS
STUDY GROUP

CALCULUS OF ELEMENTARY FUNCTIONS
Part I
Teacher's Commentary

(Revised Edition)

SE 022 992

CALCULUS OF ELEMENTARY FUNCTIONS

Part I

Teacher's Commentary

(Revised Edition)

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Teacher's Commentary

FOREWORD

Here are some comments and a course outline from the Advanced Placement Calculus AB Syllabus of the College Entrance Examination Board.

This course is intended for students who have a thorough knowledge of college preparatory mathematics, including algebra, axiomatic geometry, trigonometry, and analytic geometry (rectangular and polar coordinates [A1], equations and graphs, lines, and conics). It does not assume that they have acquired sound understanding of the theory of elementary functions. The development of this understanding has first priority.

A course in elementary functions and introductory calculus can be arranged in many ways, and the arrangement suggested in the following course outline for Calculus AB is not intended to be prescriptive. [The location of a topic in the SMSG text is indicated inside brackets, where C refers to Chapter and A to Appendix.] In this version the special functions are first studied in some detail with the aid of calculus, which is introduced intuitively, and later [C8] the general techniques of calculus are developed and applied to a wide class of functions.

1. Polynomial Functions
   - a. Definition of polynomial functions; function notation [C1,A1]
   - b. Algebra of polynomials (degree, division algorithm, remainder theorem, factor theorem, existence and number of roots, location of rational roots, approximation of irrational roots) [C1,A2]
   - c. Derivative (slope of the tangent line) [C2,4,6,8]
   - d. Applications of derivatives
     - (1) Graphs (increasing and decreasing functions, relative maximum and minimum points, concavity, and points of inflection) [C2,4,6,8]
     - (2) Extreme value problems [C2,4,6,8]
     - (3) Velocity and acceleration of a particle moving along a line [C2]
   - e. Antiderivative
     - (1) Distance and velocity from acceleration with initial conditions [C7,9]
     - (2) Polynomials as solutions of $y^{(n)} = 0$ (nth derivative identically zero) [C9]
2. Sine and Cosine Functions
   - a. Definition, fundamental identities, addition formulas [C3]
   - b. Graphs and periodicity of $A \sin (bx + c)$ and $A \cos (bx + c)$ [C3]
   - c. Derivatives of $\sin x$ and $\cos x$ [C4]

d. Derivatives of $\sin(bx + c)$ and $\cos(bx + c)$ [C4]

e. Linear approximation of $\sin x$ near $x = 0$ [C4]

f. Polynomial approximations of $\sin x$ and $\cos x$ [C4]

g. Antiderivatives [C7,9]

h. Solutions of $y'' = -k^2y$; simple harmonic motion [C9]

3. Exponential and Logarithmic Functions

a. The functions a^x and $\log_a x$ (for $a > 0$, $a \neq 1$, and $x > 0$); properties, graphs; their inverse relationship [C5]

b. The number e such that $\lim\limits_{n \to \infty} (1 + \frac{1}{n})^n = e$ and

$$\lim_{x \to 0} \frac{e^x - 1}{x} = 1 \text{ [C5,6]}$$

c. The derivatives of the exponential functions e^x and Ae^{kx} [C6] and of the logarithmic function $\ln x$ (i.e., $\log e^x$)

d. Solutions of $y' = ky$ and of $y' = cx^{-1}$; applications to growth and decay [C7,9]

e. Polynomial approximations of e^x and $\ln(1 + x)$ [C7,9]

4. Area, Average Value, and the Definite Integral

a. Concepts of area and average (mean) ordinate [C7]

b. Approximations; by inscribed and circumscribed rectangles and by trapezoids [C7]

c. The definite integral; definition and properties [C7]

d. The fundamental theorem [C7]

e. Calculation of areas and average values for polynomial, sine, cosine, and exponential functions by the fundamental theorem [C7]

5. Calculus of More General Functions

a. The function concept; algebra of functions: sum, product, quotient, composite, inverse [A1,C8]

b. Limits of functions; statement of properties [A6]

c. Continuity [A7]

d. Definition of the derivative [C2,8]

e. Derivative of sum, product, quotient (including $\tan x$) [C8]

f. Derivative of a composite function (chain rule) [C8]*

g. Derivative of an implicitly defined function; logarithmic differentiation [C8]

h. Derivative of a rational power of a function [C5,8]

i. Derivative of the inverse of a function (including $\arcsin x$ and $\arctan x$) [C8]

j. Integration by substitution [C9]

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*See Teacher's Commentary for Section 4-3.

k. Rolle's theorem; mean value theorem [C8]

ℓ. Applications of the derivative

(1) Slope of a curve [C2,4,6,8]

(2) Average and instantaneous rates of change [C2,4,6,8]

(3) Maximum and minimum values, both relative and absolute [C2,4,6,8]

(4) Discussion and sketching of curves (including such functions as $e^{-x} \sin x$ and $|f(x)|$) [C6,8]

(5) Related rates of change [C8]

m. Applications of the integral

(1) Average (mean) value of a function on an interval [C8]

(2) Areas between curves [C7]

(3) Volumes of simple solids of revolution [C9]

(4) Use of integration and inequalities to get polynomial approximations to $\sin x$, $\cos x$, e^{-x}, $\ln(1 + x)$ [C9].

(5) Interpretation of $\ln x$ as area under the graph of $y = x^{-1}$ [C7,9]

We have attempted to write the text -- not for the mathematician looking over our shoulder -- but for the student sitting before us.

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Teacher's Commentary

Chapter 1

POLYNOMIAL FUNCTIONS

We have intentionally avoided a general discussion of functions in favor of a more concrete beginning with polynomial functions.

Since the concept of a function is basic in the study of calculus, the student should have a clear understanding of the concept as well as related matters such as functional notation, operations on functions (e.g., composition and inversion). We shall consider these matters as we discuss special classes of functions such as polynomial functions, rational functions, the circular (trigonometric) functions, exponential and logarithmic functions, etc.

We have intentionally not given a formal definition of function. The precise definition of function can be formulated in many ways: as a set of ordered pairs (usually, ordered pairs of numbers), as an association or correspondence between two sets, etc. But no matter what definition we choose for a function, three things are required: a set called its domain, a set called its range, and a way of selecting a member of the range for each member of the domain.

For example, we could define a function as an association between elements of two sets.

> If with each element of a set A there is associated exactly one element of a set B, then this association is called a <u>function from A to B</u>. The set A is called the <u>domain</u> of the function, and the set C of all members of B assigned to members of A by the function is called the <u>range</u> of the function.

In this text we shall be mostly concerned with functions whose domains are subsets of real numbers and whose ranges are also subsets of real numbers. More complicated functions (like "vector valued functions") may be built from these.

The range C may be the whole set B, in which case the function is called an <u>onto</u> function or it may be a proper subset of B. In any case, we generally take for B the whole set of reals, because a function is usually specified before its range is considered.

We follow the common practice of representing a function by the letter f (other letters such as F, g, h, ϕ, etc., will also be used). If x is an element of the domain of a function f, then f(x) denotes the element of the range which f associates with x. (Read for f(x) "the value of the function f at x", or simply "f at x", or "f of x.") An arrow is used to suggest the association of f(x) with x:

$$f: x \to f(x)$$

(read "f takes x into f(x)"). This notation tells us nothing about the function f or the element x; it is merely a symbolic description of the relation between x and f(x).

As mentioned earlier, a function is not completely defined unless the domain is specified. If no other information is given, it is a convenient practice, especially when dealing with a function defined by a formula, to assume that the domain includes all real numbers for which the formula describes a real number. For example, if a domain is not specified for the function $f : x \to \frac{2x}{x^2 - 9}$, then the domain is assumed to be the set of all real numbers except 3 and -3. Similarly, if g is a function such that $g(x) = \sqrt{4 - x^2}$, we assume, in the absence of any other information, that the domain is the set of all real numbers x from -2 to 2 inclusive.

While we have not troubled the student with this, we note here that two functions f and g are identical if and only if they have the same domain and $f(x) = g(x)$ for each x in their domain.

We believe that the graph of a function is perhaps its most intuitively illuminating representation since it conveys important information about the function at a glance. We invite you to tell us where more graphs would be helpful.

We assume that the student knows that not every curve is the graph of a function. In particular, our definition of function requires that a function map each element of its domain onto only one element of its range. In terms of points of a graph, this means that the graph of a function does not contain the points (x_1,y_1) and (x_1,y_2) if $y_1 \neq y_2$; i.e., two points having the same abscissa but different ordinates. This is the basis for the "vertical line test": if in the xy-plane we imagine all possible lines which are parallel to the y-axis, and if any of these lines cuts the graph in more than one point, then the graph represents a relation which is not a function. Conversely, if every line parallel to the y-axis intersects a graph in at most one point, then the graph is that of a function.

2

Throughout this discussion we have used the letters x and y to represent elements of sets. Specifically, if f is the function

$$f : x \to y = f(x),$$

then x represents an element (unspecified) in the domain of f, and y represents the corresponding element in the range of f. In many textbooks x and y are called variables, and since a particular value of y in the range depends upon a particular choice of x in the domain, x is called the <u>independent variable</u> and y the <u>dependent variable</u>. The functional relationship is then described by saying that "y is a function of x." While this language is not frequently used in this text, we introduce it in Examples 1-1a and b since it is not uncommon and should be understood by the student.

If we avoid a general discussion of functions in the text, we also avoid a general discussion of polynomials (in favor of our concrete beginning with polynomial functions.) We mention properties of polynomials along the way as they relate to our central discussion of the behavior (and in Chapter 2 the calculus) of polynomial functions.

It is common practice to use the word "polynomial" when one means "polynomial function." Usually when we say polynomial we mean a mathematical expression in a particular form. For a mathematical phrase to be a polynomial it must satisfy two conditions:

(a) it must be formed from members of a particular set (initially in this text the set consisting of the real numbers and variables);

(b) it must be formed using no indicated operations other than addition, subtraction, or multiplication.

For your convenience we list a few phrases which are polynomials and some examples which are not polynomials.

Some Polynomials		Some Nonpolynomials	
(1)	t	(6)	$\sqrt{x} - .07$
(2)	$(u - \sqrt{3})(u + \sqrt{3})$	(7)	$\frac{3}{y - 5}$
(3)	$\sqrt{2}\, x^5 - \pi xy + \sqrt{7}$	(8)	$\frac{x^2 - 4}{x - 2}$
(4)	98	(9)	$\sqrt{(u - 4)^2}$
(5)	$\frac{2z^2}{3} - \frac{5a^3b^2c}{4}$	(10)	$\frac{(x + 2)(x^2 + 1)}{x^2 + 1}$

As is the case with the set of integers the set of polynomials is closed under addition, subtraction, and multiplication, but not under division. We observe that in (5) the rational numbers $\frac{2}{3}$ and $\frac{5}{4}$ are multiplied by other expressions and no division is involved. We say that (5) is a polynomial over the rationals or reals, while (3) is a polynomial over the reals. We shall agree to refer to expressions such as (7), (8) and (10) as rational expressions (suggested by the analogy with rational numbers). Even though (10) is the same as the polynomial $x + 2$ over the reals and (8) is not, division is indicated in each case (in violation of condition (b) of our polynomial form requirements).

Of interest to us are polynomials of the form

$$a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n,$$

where $a_0, a_1, \ldots, a_n$, are real numbers, $a_n \neq 0$, and n is a non-negative integer. We are especially concerned with functions defined by such polynomials. While we introduce polynomial functions in Section 1-1, we do not discuss the number n as the degree of the polynomial or polynomial function until Section 1-3. The degree could be included in the definition of polynomial function if you wish. A <u>polynomial function</u> of degree n, where n is a positive integer or zero, is an association

$$f_n : x \to a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0, \quad a_n \neq 0$$

where the domain is the set R of all real numbers, and the range is the set (or a subset) of real numbers

$$\{y : y = f(x), \ x \in R\}.$$

While the coefficients a_i $(i = 0, 1, \ldots, n)$ in general stand for any real numbers, in our examples and exercises they will usually represent integers. The student should understand that we could extend the domain and range of polynomial functions and the coefficients of $f(x)$ to the complex number system.

We call attention to the fact that the degree of a polynomial function is uniquely defined. That is, if for all real x a given polynomial function can be expressed as

$$x \to a_nx^n + a_{n-1}x^{n-1} + \ldots + a_1x + a_0$$

and also as

$$x \to b_m x^m + b_{m-1}x^{m-1} + \ldots + b_1 x + b_0,$$

then n must equal m, and the corresponding coefficients must also be equal,

$$a_i = b_i \quad (i = 0, 1, 2, \ldots, n).$$

We do not prove this, but we state it for the sake of completeness before the end of the chapter.

If the degree of a polynomial function is 0, then the function is

$$f : x \to a_0, \; a_0 \neq 0,$$

a constant function. In the text we find it useful to consider the special constant function $f : x \to 0$ as a polynomial function, with $f(x)$ called the zero polynomial. The zero polynomial has no degree and is not a polynomial of degree zero. We summarize:

$f : x \to a_0$, $a_0 \neq 0$, is a polynomial function of degree 0;
$f : x \to 0$ is the 0 polynomial, to which we assign no degree.

In this chapter we hope to give the student (what may be <u>for him</u>) a first introduction to the graphs of polynomial functions of degree greater than 2, and, incidentally, to reinforce the technique of synthetic substitution. Plotting a large number of points is not a very efficient way to obtain the graph of a polynomial function, but we believe that it is a helpful first step leading ultimately to proficiency in sketching graphs by means of intercepts, maximum and minimum points, and points of inflection, to be developed in Chapter 2.

The continuity of polynomial functions is assumed, but we feel that teachers should recognize the importance of this concept and be able to satisfy students on an intuitive level that the graph of a polynomial function contains no holes or breaks. The simple but tedious expedient of evaluating $f(x)$ for any suggested real number $x = c$, and also for values of x near c, should convince students of the reasonableness of the assumption.

1. $g : d \to V = \frac{4}{3}\pi\left(\frac{d}{2}\right)^3 = \frac{\pi}{6} d^3$.

2. (a) 384

   (b) 384

   (c) 384 ft.

   (d) 384 ft.

   (e) The answers are the same. After 4 seconds the pellet reaches a height of 384 feet on the way up. After 6 seconds, on the way down, the pellet again reaches a height of 384 feet.

   (f) 10 seconds

   (g) 5 seconds

   (h) 400 feet

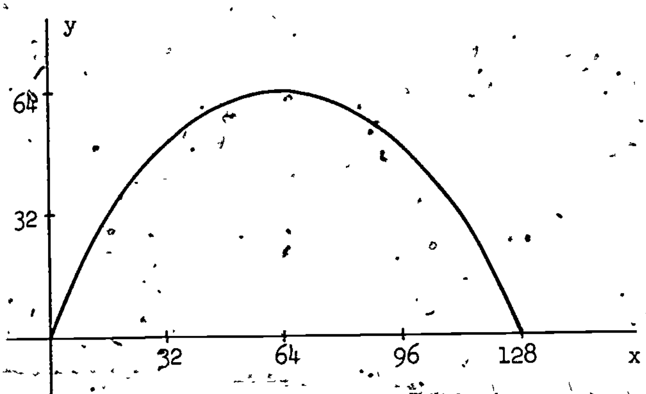
3. (a) No

   (b) parabolic

   (c) $y = 2x - \frac{x^2}{64}$

   (d) a parabola

   (e)



   (f) $y = 0$ when $x = 0$ or $x = 128$

   (g) $t = 4$ when $x = 128$

   (h) after 4 seconds

   (i) 128 feet

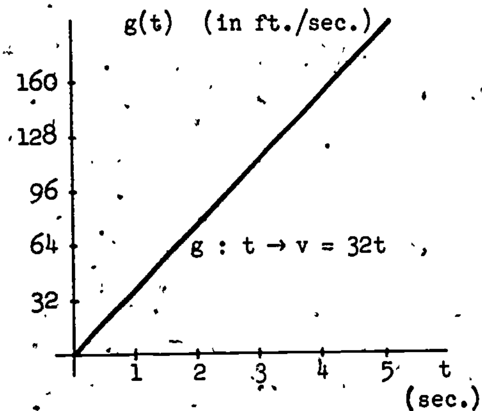
Solutions Exercises 1-2

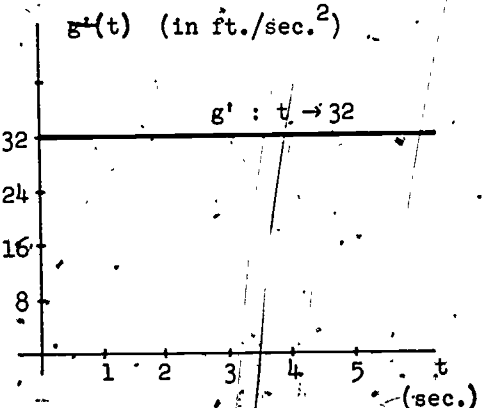
1. (a) 2

   (b) mi./hr.

   (c) They are the same.

2. (a)





   (b) $g' : t \to 32$

   (c) $\frac{\text{ft./sec.}}{\text{sec.}}$ or $\frac{\text{ft.}}{\text{sec.}^2}$

   (d) They should be the same: $\frac{\text{ft.}}{\text{sec.}^2}$

   (e) The ratio "feet per second per second" is usually associated with acceleration. In this case 32 ft./sec.^{2} refers to the acceleration due to the force of gravity at sea level. In this early example we neglect the negative sign with its connotation of downward pull. Similarly we temporarily postpone the notion of "negative velocity" or the distinction of speed as the absolute value of velocity.

3. Since the velocity v ft./sec. is directly porportional to the time t sec, we can write $v = mt$, where m is the constant of proportionality. If $v = 48$ when $t = \frac{3}{2}$, we get $m = 32$; consequently we obtain the linear function $g : t \to v = 32t$. At impact when $t = \frac{9}{2}$, $v = g(\frac{9}{2}) = 144$; therefore, the impact velocity is 144 ft./sec.

4. (a) $= 1$

(b) $= 3$

(c) $= -1$

(d) $= 2$

(e) $= 2$

5. (a) slope $= 3$

(b) slope $= -2$

(c) slope $= -\frac{1}{2}$

(d) slope $= \frac{4}{3}$

6. (a) $f : x \to -2x + 6$

(b) $f : x \to -2x - 7$

(c) $f : x \to -2x + 7$

(d) $f : x \to -2x + 13$

7. (a) slope $= -7$

(b) slope $= 6$

(c) slope $= 2$

(d) slope $= -2$

8. (a) $x \to 3x - 2$

(b) $x \to -2x - 10$

(c) no function

(d) $x \to 4$

9. The graph of f is a line with slope 3. Hence the slope of g is the number 3, so that $g(x) = 3x + b$, for some as yet unknown b. Since $g(-2) = 1$, this implies that $1 = 3(-2) + b$, $b = 7$, and thus $g : x \to 3x + 7$.

10. (a) $x \to -3x + 5$

(b) $x \to -3x - 3$

(c) $x \to -3x + 8$

(d) $x \to -3x + 13$

11. (a) $f(3) = 5$

(b) $f(3) = -3$

(c) $f(3) = 4$

12. Yes. The slope of the line through P and Q is -2 and the slope of the line through P and S is -2. Two lines through the same point with the same slope coincide. (Distance arguments could also be used.)

13. (a) $(100.1 - 100)\left(\frac{39 - 25}{101 - 100}\right) + 25 = .1(14) + 25 = 26.4$

$f(100.1) = 26.4$

(b) $.3(14) + 25 = 4.2 + 25 = 29.2 \quad f(100.3) = 29.2$

(c) $f(101.7) = 48.8$

(d) $f(99.7) = 20.8$

14. (a) $f(53.3) = -44(.3) + 25 = 11.8$

(b) $f(53.8) = -10.2$

(c) $f(54.4) = -36.6$

(d) $f(52.6) = 42.6$

15. $\begin{cases} 2x + 7y + 1 = 0 \\ x - 2y + 8 = 0 \end{cases}$

$\begin{cases} 2x + 7y + 1 = 0 \\ 2x - 4y + 16 = 0 \end{cases}$

$11y = 15$

$\begin{cases} y = \frac{15}{11} \\ x = -\frac{58}{11} \end{cases}$

$x - 3y + 4 = 0$

Slope $= \frac{1}{3}$

$y = \frac{1}{3}x + b$

$\frac{15}{11} = \frac{1}{3}\left(-\frac{58}{11}\right) + b$

$\frac{103}{33} = b$

$y = \frac{x}{3} + \frac{103}{33}$ or

$11x - 33y + 103 = 0$

16. The slopes of the lines AB and CD are $\frac{1}{4}$ and the slopes of the lines AD and BC are $-\frac{3}{2}$. Since the opposite sides are parallel (have the same slope), ABCD is a parallelogram.

17. (a) $C(4,8)$

(b) $C(5,-11)$

18. $f : x \to 2x - 1$

$f(t + 1) = 2(t + 1) - 1 = 2t + 1$

Therefore, $P(t + 1, 2t + 1)$ is on the graph of f.

19. (a) When $t - 1 = 0$, $t = 1$; $f(t - 1) = f(0) = 3(1) + 1 = 4$;

When $t - 1 = 8$, $t = 9$; $f(t - 1) = f(8) = 3(9) + 1 = 28$;

(b) When $t = 1$, $f(t - 1) = f(0) = 1^2 + 1 = 2$;

$t = 9$, $f(t - 1) = f(8) = 9^2 + 1 = 82$.

20. -40

21. We have $f(x_1) = mx_1 + b$, $f(x_2) = mx_2 + b$.

Thus, $f(x_1) - f(x_2) = mx_1 + b - (mx_2 + b)$

$= mx_1 - mx_2$

$= m(x_1 - x_2)$.

Since $m < 0$ and $x_1 < x_2$ or $x_1 - x_2 < 0$, we get $m(x_1 - x_2) > 0$.

Therefore, $f(x_1) - f(x_2) > 0$ or $f(x_1) > f(x_2)$.

22. $\mu = \frac{1}{m}$

23. $g : x \to \frac{1}{m}x - \frac{b}{m}$

24. $y = -\frac{1}{m}x + b$

25. $f(2x) = 4x^2 - 4x$

Solutions Exercises 1-3

Most of Exercises 1-3 are intended as review and need not be assigned. If a student encounters difficulty with the calculus of quadratics in Chapter 2, some of these problems (done by familiar algebraic techniques) might serve as plausibility tests for new concepts and methods.

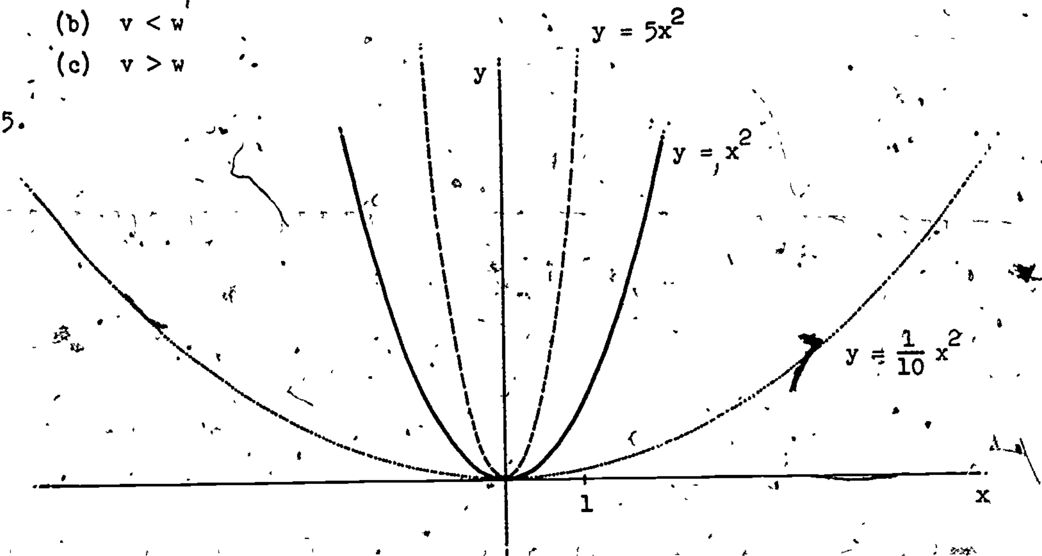
1. (a) constant
   (b) linear
   (c) linear
   (d) quadratic
   (e) quadratic
   (f) quadratic

2. (a) 160 ft./sec.
   (b) 320 ft./sec.

3. (a) 3 sec.
   (b) 1600 ft.

4. (a) $v < w$
   (b) $v < w$
   (c) $v > w$

5.



6. The points (p,q) and $(-p,q)$ are the same distance ($|q|$ units) from the X-axis and on the same side of it; they are the same distance ($|p|$ units) from the Y-axis and on opposite sides of it.

7. (a) 512 ft.
   (b) 576 ft.
   (c) 576 ft. (In fact the Time-Life Building is 587 ft. high and a 47th story window is about 576 ft. above the ground.)
   (d) $f(4) = 320$
   (e) 320 ft.
   (f) $t = 6$
   (g) 6 seconds

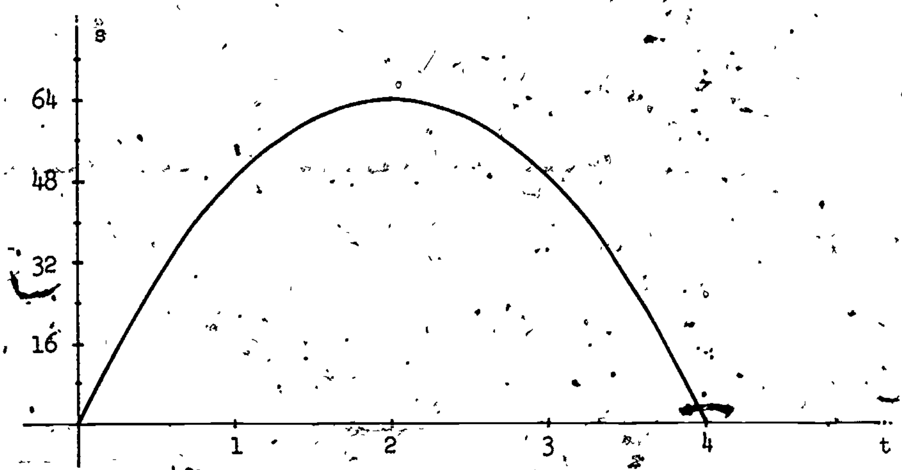
8. (a) 400 ft.
   (b) 15 seconds

9. (a) $a = -16$, $c = 784$
   (b) 7 seconds

10. (0,c)
    upward
    lowest
    parabola
    Y-axis
    above

11. (a) 8 seconds
    (b) 1046 feet

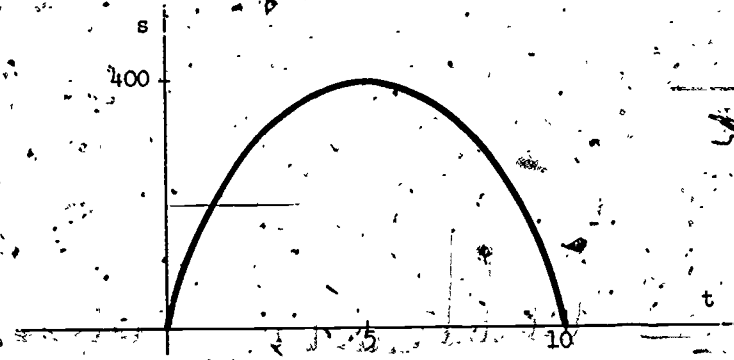
12. 640 ft.
    (a) 48
    (b) 64
    (c) 48 ft. 0 ft.
    (d)



(e) Straight up and straight down.

(f) parabola

13.



14. -1 and 0 or 0 and 1

15. (a) 5 seconds

(b) 48 ft. for both

(c) after 4 seconds

(d) after 2 seconds

(e) 64 feet

In our idealized problem (No. 15) we say "close to the edge" and "nearly vertical path" to allow the ball to miss the top of the building in its "vertical" descent all the way to the ground. Similarly, when we launch a projectile vertically from a platform we shall assume that the platform collapses on take-off, allowing the projectile to descend all the way to the ground.

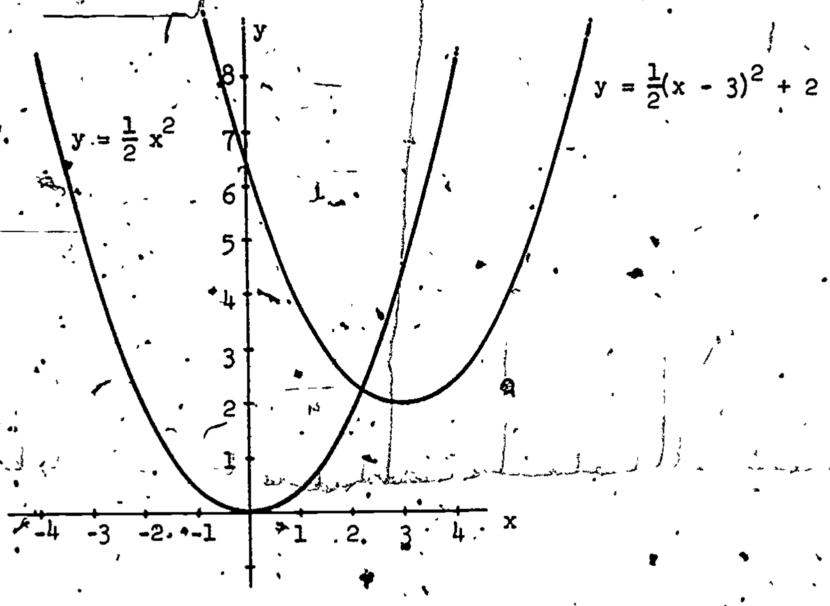
It should be emphasized that while we picture the motion function as a parabola, we think of the physical motion of the projectile itself as vertical. Accordingly, we repeat the phrases "straight up" and "straight down."

16. (a) For $u = 4$, $v = w$
For $u < 4$ or $u > 4$, $v > w$.

(b) For $u = 0$, $v = w$
For $u > 0$, $v < w$
For $u < 0$, $v > w$

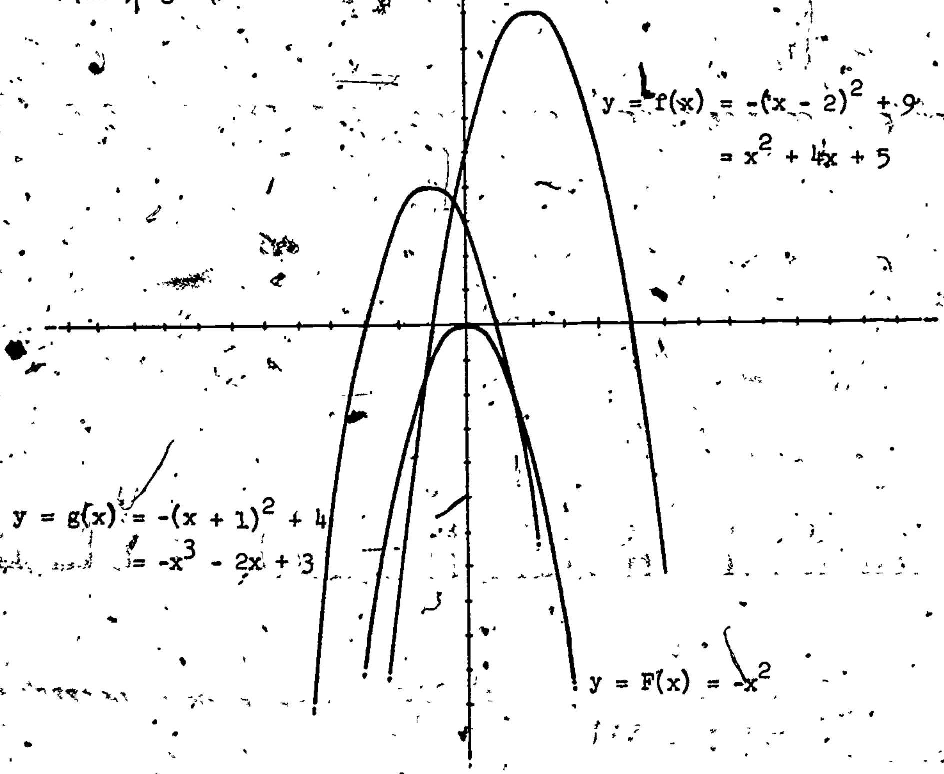
	$x \to \frac{1}{2}x^2$	$x \to \frac{1}{2}(x-3)^2 + 2$
(a)	parabola	parabola
(b)	upward	upward
(c)	$x = 0$	$x = 3$
(d)	(0,0)	(3,0)
(e)	minimum	minimum

18.



19.

(a)	(b)
(i)	F
(ii)	f
(iii)	g



20. (a) ap^2

(b) We have

$$a((p + h) - h)^2 + k = a(p + h - h)^2 + k$$
$$= ap^2 + k.$$

From part (a) we know that $ap^2 = q$.

Therefore, $g(p + h) = q + k$.

(c) $q + k$

(d) $(0,0)$

(e) (h,k)

We summarize important properties of the graph of $x \to a(x - h)^2 + k$.

(1) The graph is congruent to the graph of $x \to ax^2$, but has a position

(a) $|h|$ units to the right or left of the graph of $x \to ax^2$ according as $h > 0$ or $h < 0$;

and

(b) $|k|$ units up or down from the graph of $x \to ax^2$ depending on whether $k > 0$ or $k < 0$.

(2) The graph is symmetric with respect to the line whose equation is $x = h$, and this line is called the axis of the parabola.

(3) The point (h,k) is an extremum point. If $a > 0$, the graph opens upward and (h,k) is the minimum point; if $a < 0$ the graph opens downward and (h,k) is the maximum point. In either case, the point (h,k) is the vertex of the parabola.

21.

vertex	equation of axis
(3,4)	$x = 3$
(3,4)	$x = 3$
(-3,0)	$x = -3$
(1,-1)	$x = 1$
(-1,2)	$x = -1$
(2,-3)	$x = 2$

22. (a) $(3,4)$ is a minimum
(b) $(3,4)$ is a maximum
(c) $(-3,0)$ is a minimum
(d) $(1,-1)$ is a maximum
(e) $(-1,2)$ is a minimum
(f) $(2,-3)$ is a minimum

23. (a) $v > w$ for all values of u

(b) $v > w$ for all values of u

(c) $u < 0$ when $v > w$

$u = 0$ when $v = w$

$u > 0$ when $v < w$

24. (a) $y = (x - 3)^2$

(b) $y = 2(x - 3)^2$

(c) $y = 2(x - 3)^2$

(d) $y = 2(x - 3)^2 + 4$

(e) $y = 2(x - 3)^2 + 4$

(f) $y = -2(x - 3)^2$

(g) $y = -2(x - 3)^2 + 4$

(h) $y = -2(x - 3)^2 + 4$

(i) $y = (x + 3)^2$

(j) $y = (x - 1)^2$

(k) $y = -\frac{1}{2}(x - 1)^2$

(ℓ) $y = -\frac{1}{2}(x - 1)^2 - 1$

(m) $y = (x + 1)^2$

(n) $y = 3(x + 1)^2$

(o) $y = 3(x + 1)^2 + 2$

(p) $y = (x - 2)^2$

(q) $y = \frac{1}{5}(x - 2)^2$

(r) $y = \frac{1}{5}(x - 2)^2$

(s) $y = \frac{1}{5}(x - 2)^2 - 3$

25. If we let $ax^2 + bx + c = 0$, then we have

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

To complete the square on the left we add $(\frac{b}{2a})^2$ to each side, and get

$$x^2 + \frac{b}{a}x + (\frac{b}{2a})^2 = -\frac{c}{a} + \frac{b^2}{4a^2}.$$

Thus, we can write

$$(x - \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a^2}.$$

For $b^2 - 4ac \geq 0$, we have

$$x - \frac{b}{2a} = \pm \frac{\sqrt{b^2 - 4ac}}{2a},$$

and conclude that

$$x_1 \text{ or } x_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Solutions Exercises 1-4

1. (a) $f(-2) = 11$, $f(1) = -1$, $f(3) = 81$

   (b) $f(-1) = 1$, $f(-3) = 85$, $f(0) = -2$, $f(2) = -20$, $f(4) = -174$

   (c) $g(\frac{1}{2}) = \frac{7}{8}$, $g(\frac{1}{3}) = \frac{8}{9}$, $g(2) = 17$

   (d) $r(\frac{3}{2}) = -\frac{21}{2}$, $r(\frac{1}{2}) = -3$, $r(-\frac{1}{2}) = \frac{25}{2}$, $r(\frac{1}{3}) = 0$, $r(2) = 0$

   (e) $s(0) = -12$, $s(1) = 2$, $s(2) = -6$, $s(3) = 0$, $s(4) = 56$

   (f) $F(\frac{1}{3}) = 5$

   (g) $G(2) = 1002$

2. $k = 3$

3. $k = 0$

4. (a) (i) 2

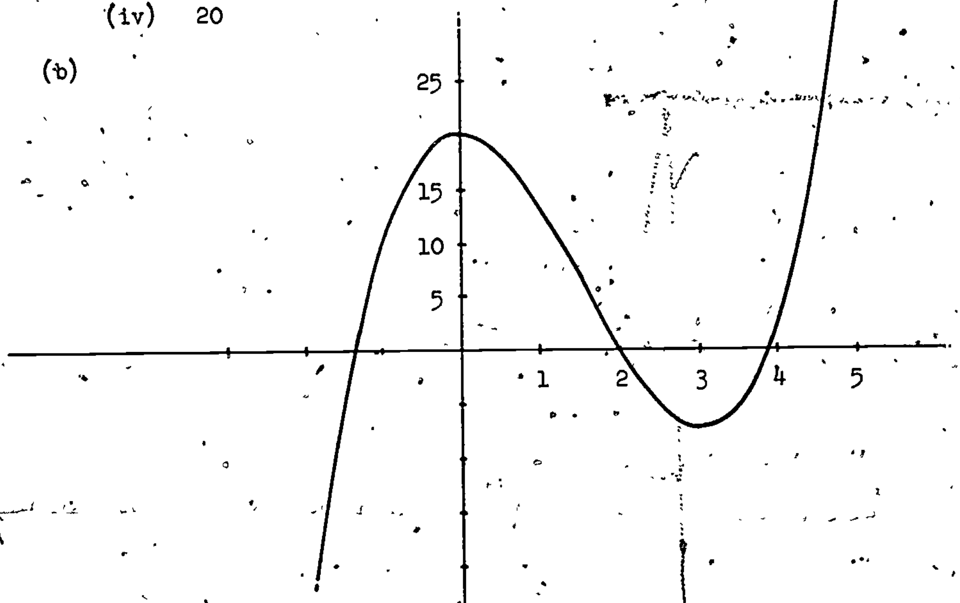
   (ii) -9

   (iii) 0

   (iv) 20

   $x \to 2x^3 - 9x^2 + 20$

   (b)



   (c) The graph of part (b) is the graph of Figure 1-4b shifted one unit to the right.

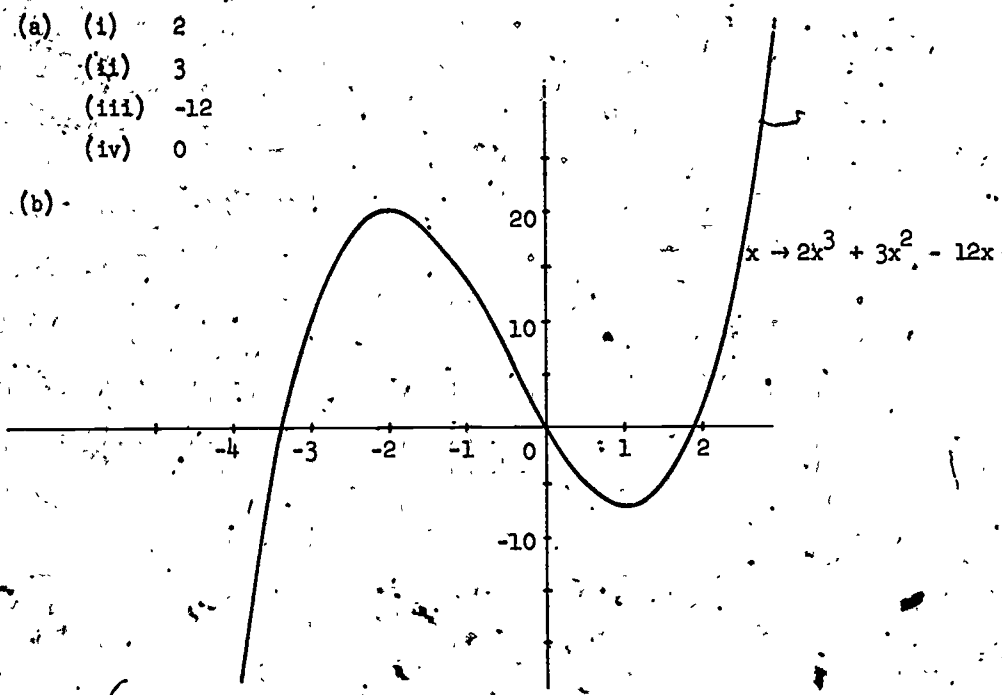
5. (a) (i) 2

(ii) 3

(iii) -12

(iv) 0

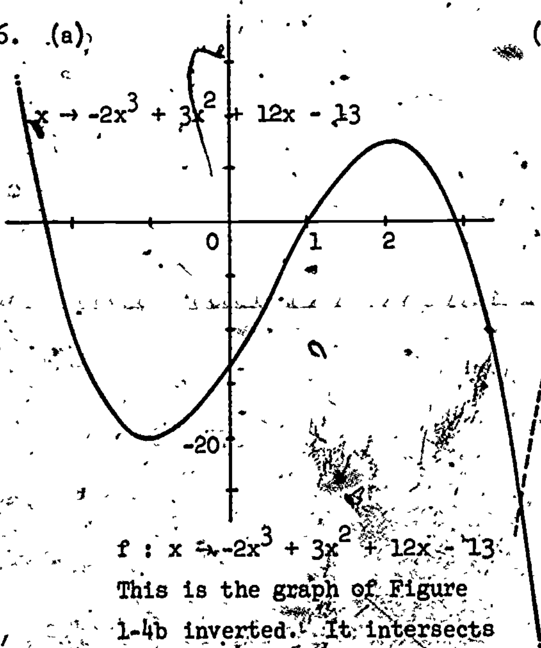
(b)



(c) The graph of part (b) is the graph of Figure 1-4b shifted one unit to the left.

Should a student discover a pattern in Nos. 4 and 5, he could be encouraged to search for reasons to support any tentative hypothesis (and promised full explanation in subsequent sections of the text).

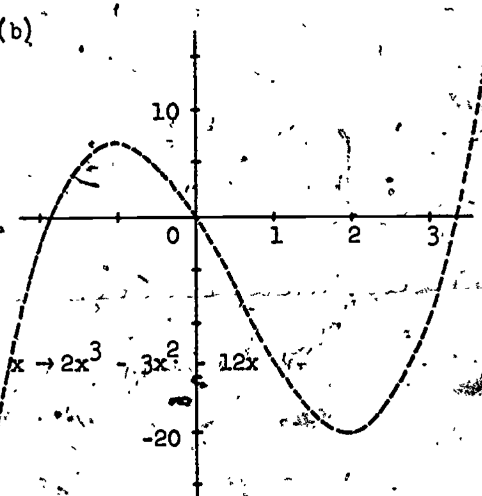
6. (a)



$f : x \to -2x^3 + 3x^2 + 12x - 13$

This is the graph of Figure 1-4b inverted. It intersects the x-axis at the same points.

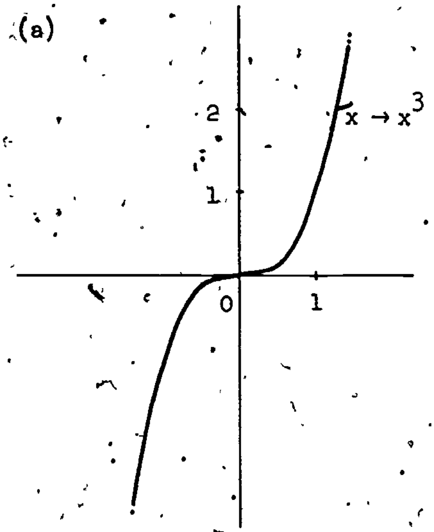
(b)



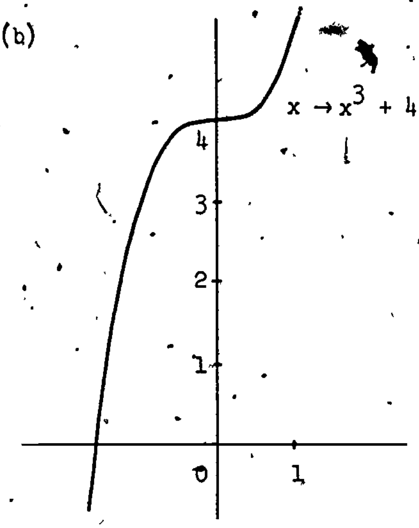
$f : x \to 2x^3 - 3x^2 - 12x$

This is the graph of Figure 1-4b moved down 13 units.

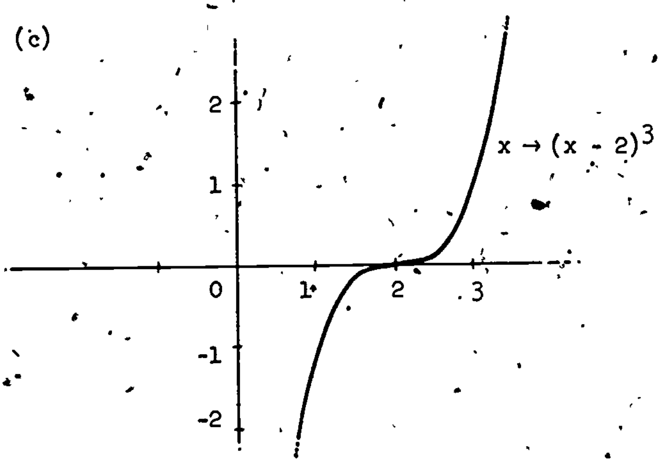
7. (a)


$x \to x^3$
2
1
0
1

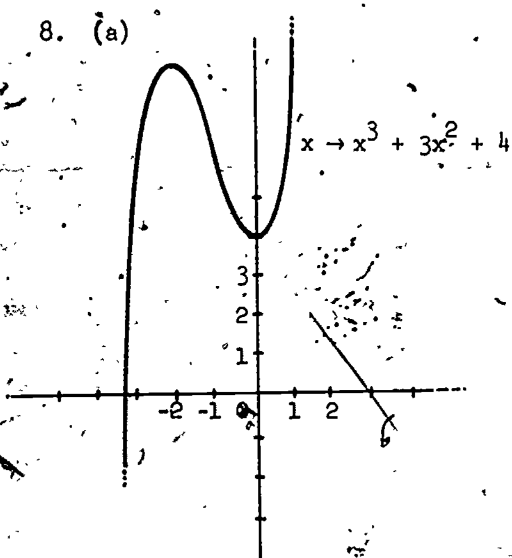
(b)


$x \to x^3 + 4$
4
3
2
1
0
1

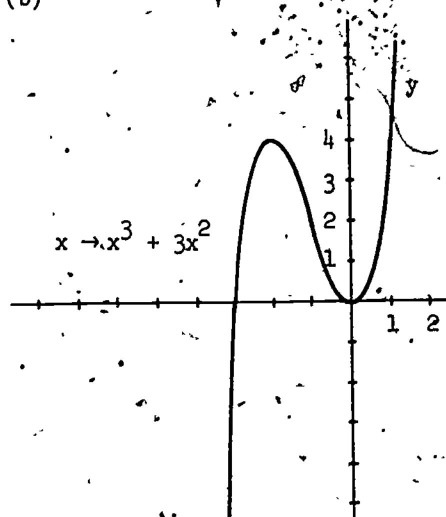
(c)


$x \to (x - 2)^3$
2
1
0
1
2
3
-1
-2

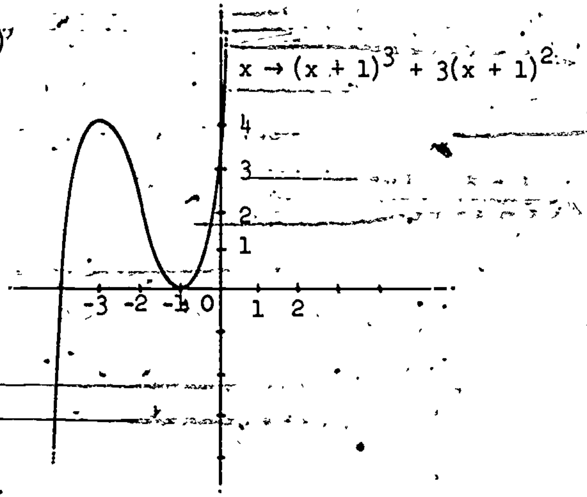
8. (a)


$x \to x^3 + 3x^2 + 4$
3
2
1
-2
-1
0
1
2

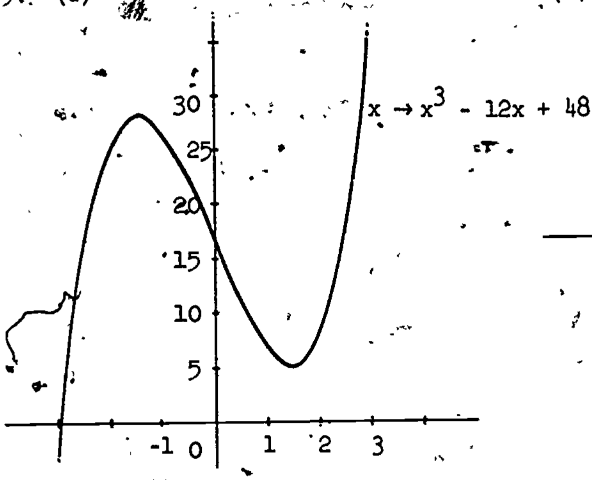
(b)


y
$x \to x^3 + 3x^2$
4
3
2
1
1
2

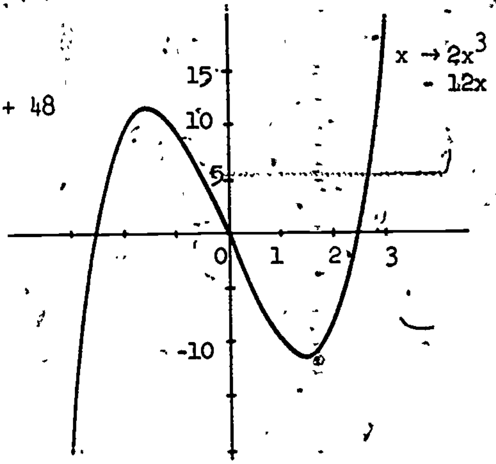
(c)


$x \to (x + 1)^3 + 3(x + 1)^2$
4
3
2
1
-3 -2 -1 0 1 2

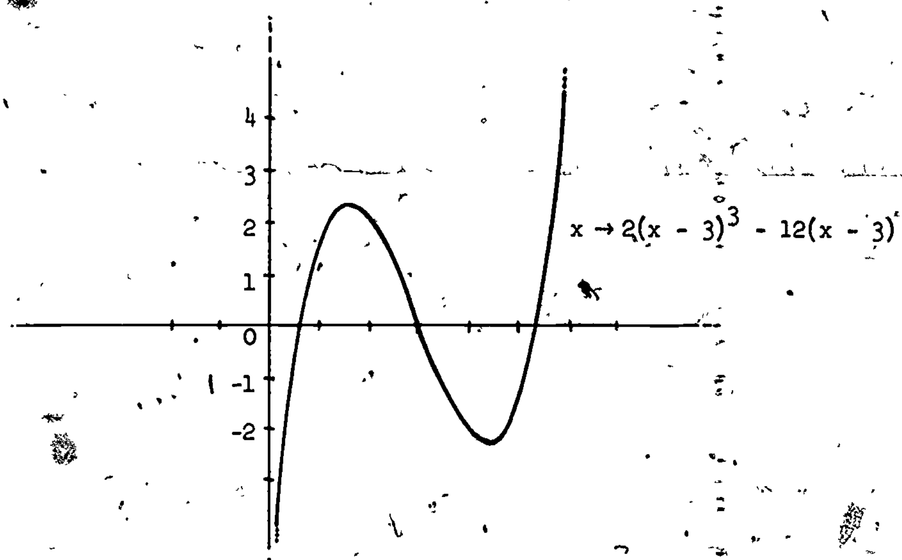
9. (a)


$x \to x^3 - 12x + 48$
30
25
20
15
10
5
-1 0 1 2 3

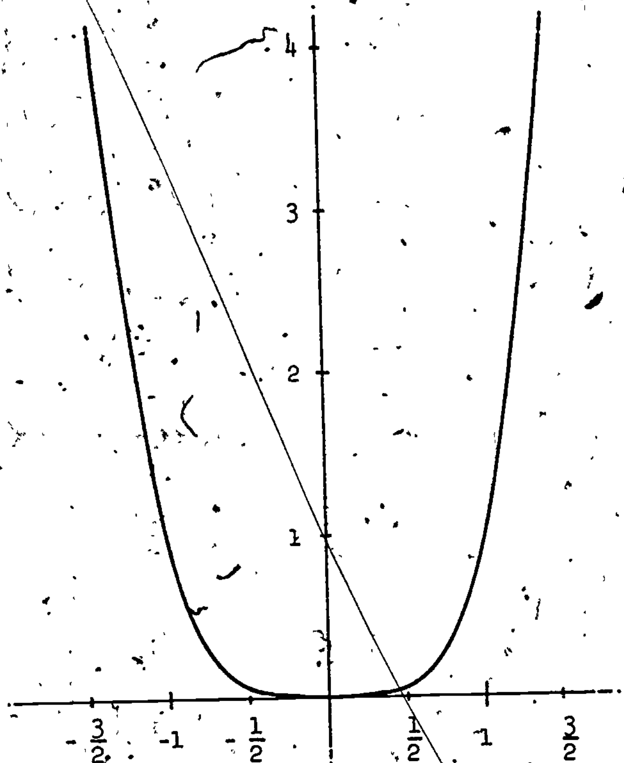
(b)


$x \to 2x^3 - 12x$
15
10
5
-10
0 1 2 3

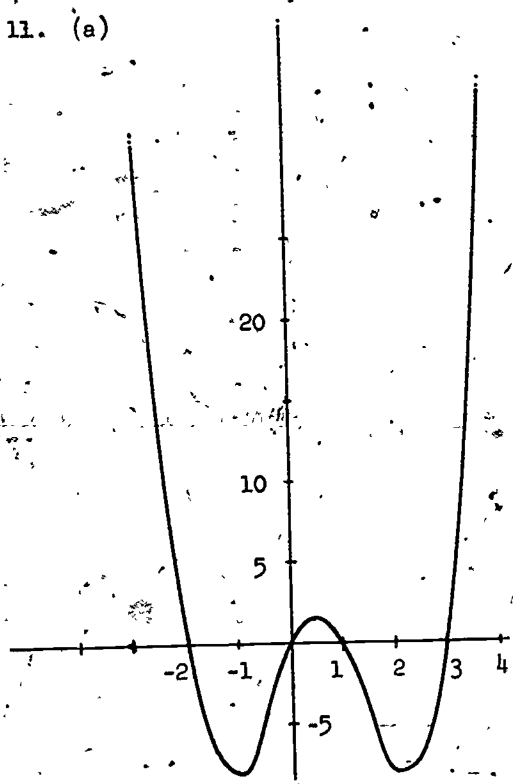
(c)


$x \to 2(x - 3)^3 - 12(x - 3)$
4
3
2
1
0
-1
-2

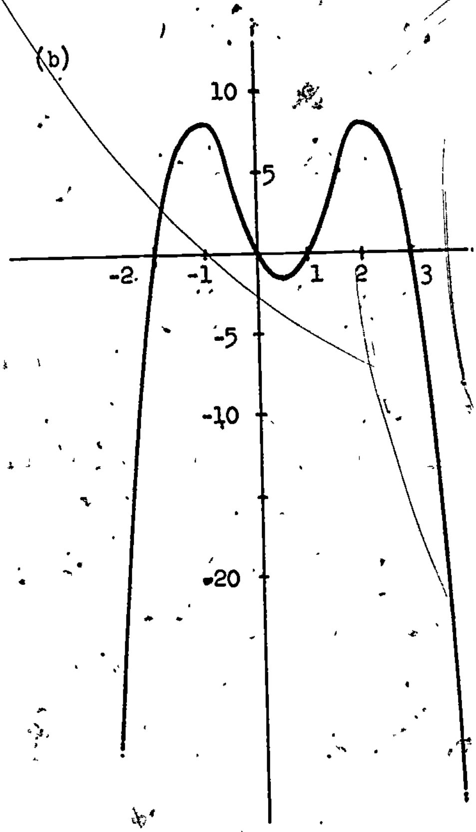
10.


4
3
2
1
$-\frac{3}{2}$ -1 $-\frac{1}{2}$ $\frac{1}{2}$ 1 $\frac{3}{2}$

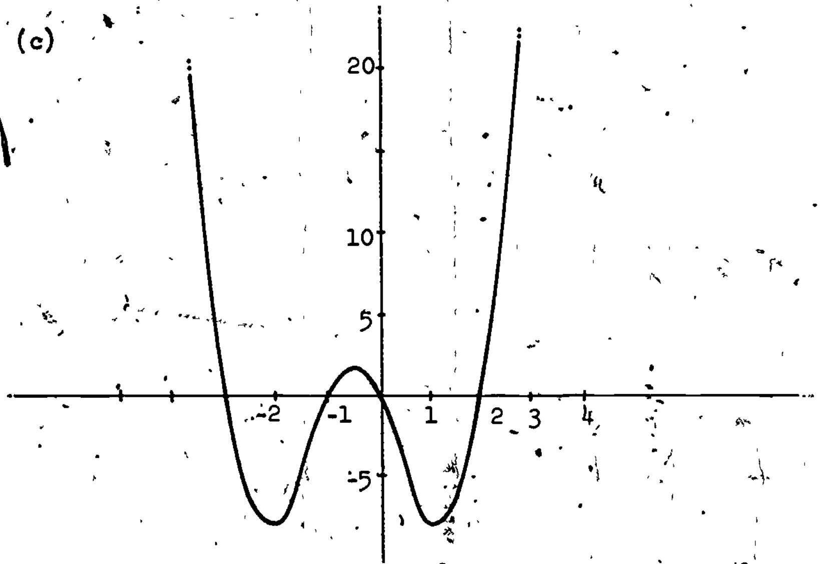
11. (a)


20
10
5
-5
-2 -1 1 2 3 4

(b)

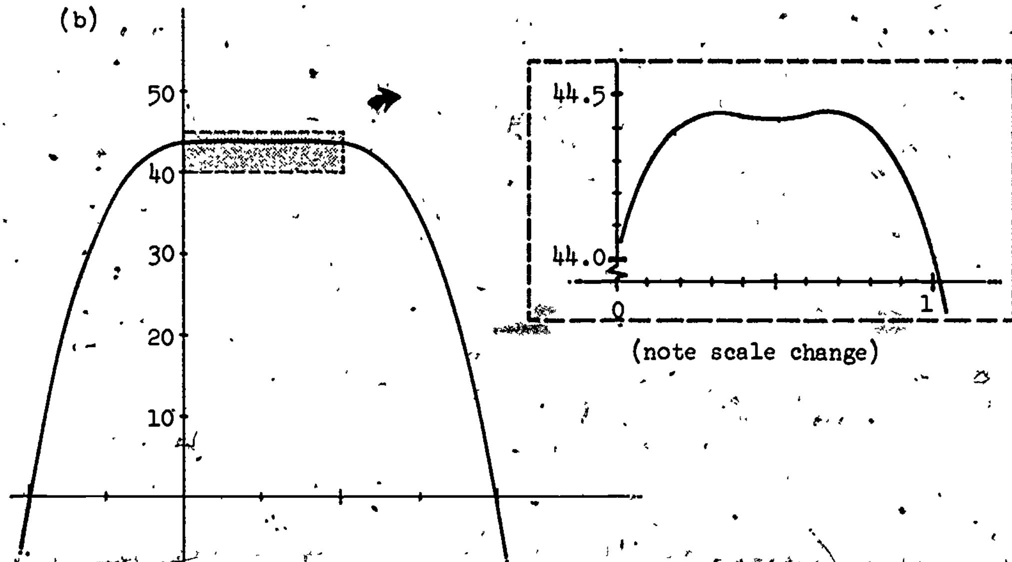

10
5
-5
-10
-20
-2 -1 1 2 3

(c)



12. (a) The maximum is not at $x = \frac{1}{2}$, but rather at $x = \frac{1}{3}$ or $x = \frac{2}{3}$; $f(\frac{1}{2}) \doteq 44\frac{7}{16}$, while $f(\frac{1}{3}) = f(\frac{2}{3}) = 44\frac{4}{9} > 44\frac{7}{16}$; $f(\frac{1}{2})$ is in fact a local minimum.

(b)



(note scale change)

13. The following table suggests that the maximum value of f occurs when $x = -\frac{1}{2}$.

x	-2	-1	0	1
f(x)	-2520	39	39	-2520

Actually $f(-\frac{1}{2}) = -39$ is a relative (local) minimum. The (global) maximum value of the function is $f(-1) = f(0) = 39$.

Solutions Exercises 1-5

1.

	$q(x)$	$f(c)$
(a)	$3x^2 + 10x + 10$	5
(b)	$x^2 + 2$	6
(c)	$-2x^3 - 3x^2 - 9x - 21$	-73
(d)	$2x^2 - 2x + 4$	0

2.

	Quotient	Remainder
(a)	$x^2 + 6x + 5$	7
(b)	$x^2 + x - 2$	0
(c)	$x^2 + 2x - 1$	1

3. $q(x)$ is of degree $n - m$.
   $r(x)$ is of degree less than m.

4. (a) $x - 2$ and $x - \frac{1}{3}$ are linear factors.
   (b) $x - 3$ is a linear factor

5. (a)

x	-3	-2	-1	0	1	2	3
f(x)	0	0	-4	-6	0	20	60

   (b) $f(x) = (x + 3)(x + 2)(x - 1)$

6.

2	1	-5	2	
2	-3	1	0	-2
2	-1	-4	6	-1
2	1	-5	2	0
2	3	-2	0	1
2	5	5	12	2
2	2	-4	0	$\frac{1}{2}$
			f(x)	x

$$f(x) = 2x^3 + x^2 - 5x + 2$$
$$= 2(x + 2)(x - 1)(x - \tfrac{1}{2})$$
$$= (x + 2)(x - 1)(2x - 1)$$

7. $f(3) = 18 - k = 9$, $k = 9$

8. If $f(x)$ is exactly divisible by $x - 3$, then $f(3) = 0$. But by substitution (direct or synthetic), $f(3) = 3k + 6$. Hence, $3k + 6 = 0$ and $k = -2$.

9. Since $f(-1) = -2a - 14 = 0$, $a = -7$. Hence $f(1) = -8$.

10. (a) The quotient is $x^4 + 2x^3 - 3x^2 - 4x + 4$.

(b) $g(1) = 0$

(c) $\alpha = 1$, $\beta = 4$, $\gamma = 4$

(d) $A = 1$, $B = 1$, $C = -5$, $D = -1$, $E = 8$, $F = -4$

11. (a) $g(x) = x^2 + 2x + 1$, $f(2) = 2$

(b) $p(x) = x + 4$, $g(2) = 9$

(c) $q(x) = 1$, $p(2) = 6$

(d) $q(2) = 1$

(e) We have

$$\begin{aligned} f(x) &= (x - 2)g(x) + f(2), \\ g(x) &= (x - 2)p(x) + g(2), \\ p(x) &= (x - 2)q(x) + p(2), \\ q(x) &= q(2). \end{aligned}$$

Therefore,

$$\begin{aligned} f(x) &= (x - 2)[(x - 2)p(x) + g(2)] + f(2) \\ &= (x - 2)\{(x - 2)[(x - 2)q(x) + p(2)] + g(2)\} + f(2) \\ &= (x - 2)\{(x - 2)[(x - 2)q(2) + p(2)] + g(2)\} + f(2). \end{aligned}$$

(f) From part (e) we have

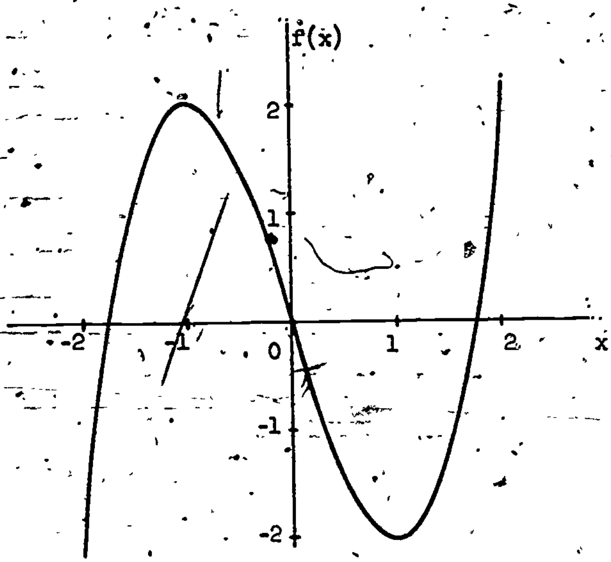
$$f(x) = q(2)(x - 2)^3 + p(2)(x - 2)^2 + g(2)(x - 2) + f(2).$$

Substituting from parts (a) through (d) we get

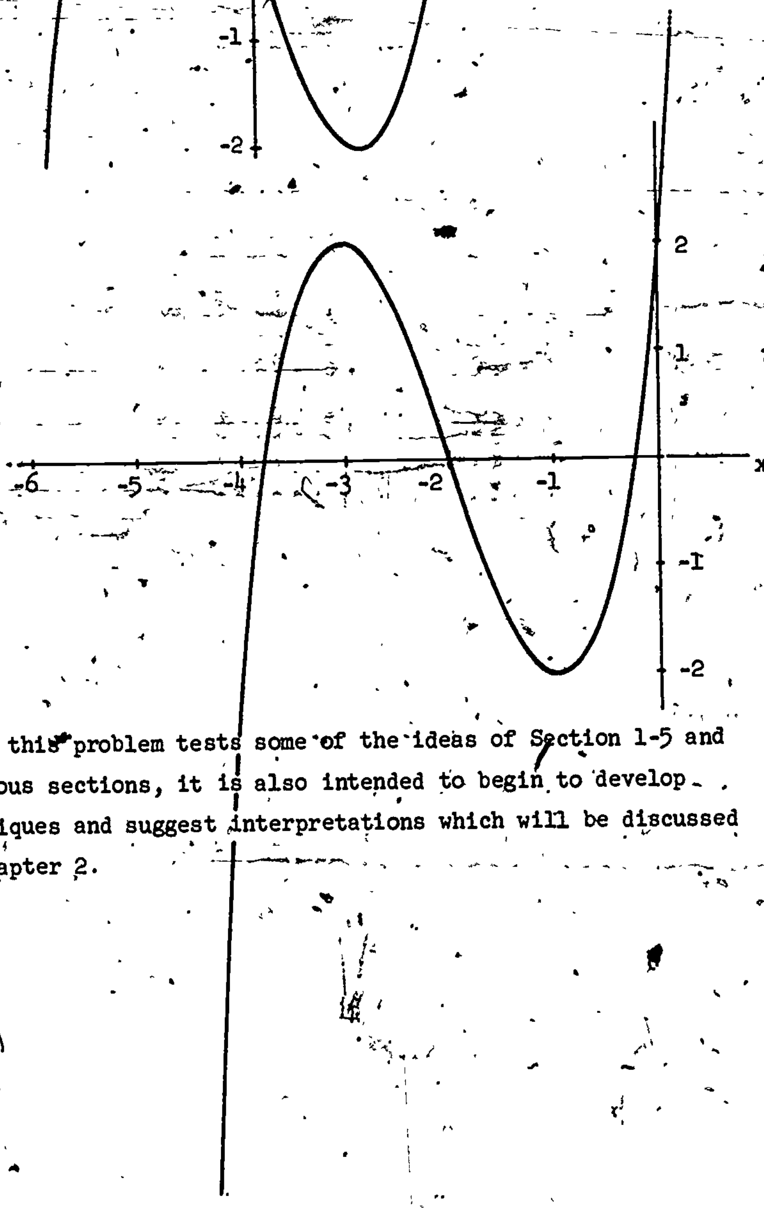
$$f(x) = (1)(x - 2)^3 + (6)(x - 2)^2 + 9(x - 2) + 2.$$

Therefore, $A = 1$, $B = 6$, $C = 9$, and $D = 2$.

(g)



(h)



While this problem tests some of the ideas of Section 1-5 and previous sections, it is also intended to begin to develop techniques and suggest interpretations which will be discussed in Chapter 2.

1. (a) $-1 < x < 0$; $1 < x < 2$; and $2 < x < 3$

   (b) $0 < x < 1$

   (c) $1 < x < 2$

   (d) $-2 < x < -1$; and two zeros such that $0 < x < 1$

   (e) $-1 < x < 0$; $1 < x \le 2$; and $2 < x < 3$

   (f) $2 < x < 3$

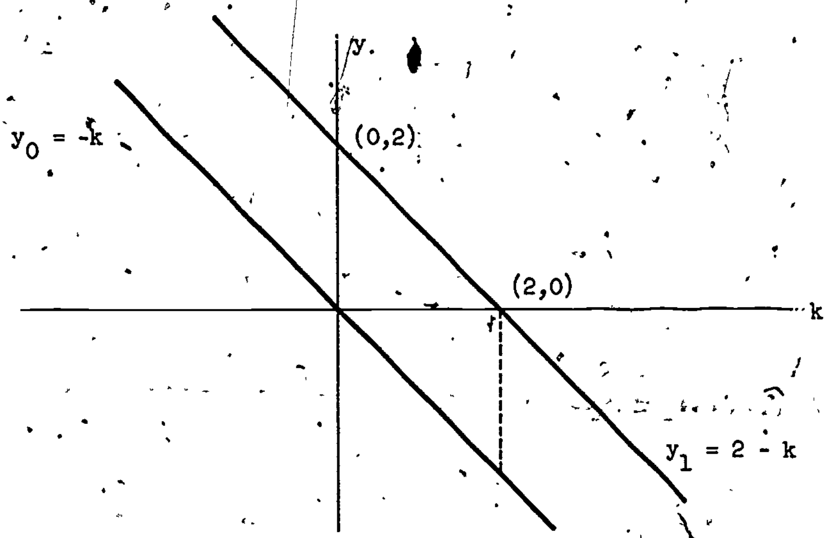
   (g) $-2 < x < -1$; $0 < x < 1$; $1 < x \le 2$; and $5 < x < 6$

2. $f(x) = x^3 - 2x^2 + 3x - k$

   (a) $f(0) = -k$ and $f(1) = 2 - k$

   For $-k$ and $2 - k$ to be different in sign, we must have $0 < k < 2$, since, if $k < 0$, both $-k > 0$ and $2 - k > 0$. If $k > 2$, both $-k < 0$ and $2 - k < 0$.

   Another way to see this is to graph $y_0 = -k$ and $y_1 = 2 - k$. A value of k for which y_0 and y_1 are both positive or both negative must be rejected. But when one y is above the k-axis and the other is below, we have a possible k-value. (See figure.)



   (b) $f(1) = 2 - k$ and $f(2) = 6 - k$. Hence $2 < k < 6$.

3. (a) $f(-\frac{3}{2}) = 0$

(b) The quotient is $2x^2 - 4x + 4$. Both (a) and (b) can be done simultaneously by synthetic substitution (division).

(c) $x = 1 \pm i$, where $i = \sqrt{-1}$

(d) Only once.

(e) One

(f) $-\frac{3}{2}$, $1 + i$, $1 - i$

4. (a) (i) $-2 < x < -1$, $x = 0$, $1 < x < 2$

(ii) $-4 < x < -3$, $x = -2$, $-1 < x < 0$

(b) $x(x^2 - 3) = x(x + \sqrt{3})(x - \sqrt{3})$

(c) (i) $-\sqrt{3}$, 0, $\sqrt{3}$

(ii) $-2 - \sqrt{3}$, -2, $-2 + \sqrt{3}$

5. $(x + 2)(x - 1)(x - 3) = x^3 - 2x^2 - 5x + 6 = 0$

6. (a) 2. It is the negative of the coefficient of x^2 in Number 5.

(b) -5. It is the same as the coefficient of x in Number 5.

(c) -6. It is the negative of the constant term in Number 5.

7. (a) $\frac{3}{2}$

(b) $-\frac{11}{2}$

(c) -3

(d) $x^3 - \frac{3}{2}x^2 - \frac{11}{2}x + 3 = 0$

(e) $(x + 2)(x - \frac{1}{2})(x - 3) = x^3 - \frac{3}{2}x^2 - \frac{11}{2}x + 3 = 0$

8. (a) $(x - r_1)(x - r_2)(x - r_3)$

$$= x^3 - (r_1 + r_2 + r_3)x^2 + (r_1r_2 + r_1r_3 + r_2r_3)x - r_1r_2r_3 = 0$$

(b) $\frac{a_2}{a_3} = -(r_1 + r_2 + r_3)$

$\frac{a_1}{a_3} = r_1r_2 + r_1r_3 + r_2r_3$

$\frac{a_0}{a_3} = -r_1r_2r_3$

9. Any 3rd-degree polynomial function with zeros -1, 1, and 4 is

$$f : x \to a_3(x^3 - 4x^2 - x + 4).$$

From this, $f(0) = 4a_3$. Since $f(0) = 12$, it follows that $4a_3 = 12$ and $a_3 = 3$. Hence, the required function is

$$f : x \to 3x^3 - 12x^2 - 3x + 12.$$

10.

	Maximum Number of Real Roots	
	Positive	Negative
(a)	2	1
(b)	2	1
(c)	1	1
(d)	1	0
(e)	0	1
(f)	0	0

Solutions Exercises 1-7

1. (a) $-\frac{1}{2}$, 2

   (b) $-\frac{1}{2}$, 0, 2

2. (a) 1, 2, 3

   (b) 0, 1, 2, 3

3. (a) No rational zeros.

   (b) 0

4. (a) $-1, \frac{1}{2}, 1$

   (b) $-1, 0, \frac{1}{2}, 1$

5. $-\frac{1}{2}, \frac{3}{2}, \frac{7}{3}$

6. $\frac{4}{3}$; $1 + \sqrt{2}$, $1 - \sqrt{2}$

7. No rational zeros.

8. $-2, -1, 2, 3$

9. $-2, 2$ (Each of these is a zero of multiplicity two. See Section A2-3.)

10. $-1, 1, 2, 3$

11. $-3, -2, -1, 1, 2$

12. $-3, \frac{5}{3}, 2 + \sqrt{3}, 2 - \sqrt{3}$

13. $x + \frac{1}{x} = n \Leftrightarrow x^2 - nx + 1 = 0$ if $x \neq 0$. The discriminant for this quadratic is $n^2 - 4$. If $|n| < 2$, $n^2 < 4$ and $n^2 - 4 < 0$, which means that the roots are imaginary.

1. $f(0.3) = -0.073$
   $f(0.4) = 0.264$

   Thus if $f(x_0) = 0$ then $0.3 < x_0 < 0.4$.

2. (a) $f(0.31) = -0.040$
   $f(0.35) = 0.093$

   Thus if $f(x_0) = 0$ then $0.31 < x_0 < 0.35$.

   (b) $f(0.33) \approx 0.026$
   $f(0.32) \approx -0.007$

   Thus if $f(x_0) = 0$ then $0.32 < x_0 < 0.33$.

   $f(0.325) \approx 0.010$

   Thus $0.320 < x_0 < 0.325$ and $x_0 \approx 0.32$ to the nearest 0.01.

3. $f(3) = -1$
   $f(4) = 13$

   Thus if $f(x_0) = 0$ then $3 < x_0 < 4$.

   $f(3.5) \approx 4.125$

   Thus $3 < x_0 < 3.5$.

   $f(3.25) = 1.141$

   Thus $3.0 < x_0 < 3.25$ and $x_0 \approx 3.0$ to the nearest 0.5.

4. Locating the zeros of $f : x \to x^3 - 2x^2 + x - 3$.

1	-2	1	-3		
1	-2	1	-3	0	Lower Bound
1	-1	0	-3	1	
1	0	1	-1	2	x_0
1	1	4	+9	3	Upper Bound
			$f(x)$	x	

   (a) $f(2.5) = 2.625$

   Thus if $f(x_0) = 0$ then $2.0 < x_0 < 2.5$.

   $f(2.25) = 0.516$

   Thus $2.0 < x_0 < 2.25$ and $x_0 \approx 2.0$ to the nearest 0.5.

(b) $f(2.125) \approx -0.311$

Thus $2.125 < x_0 < 2.25$ and $x_0 \approx 2.2$ to the nearest 0.1.

5. Locating zeros of $f : x \to x^3 + x - 3$.

1	0	1	-3		
1	0	1	-3	0	
1	1	2	-1	1	x_0
1	2	5	7	2	Upper Bound
1	0	1	-3	0	
1	-1	2	-5	-1	Lower Bound
			$f(x)$	x	

(a) $f(1) = -1$

$f(2) = 7$

$f(1.5) = 1.875$

Thus, if $f(x_0) = 0$ then $1.0 < x_0 < 1.5$.

$f(1.25) \approx 0.203$

Thus $1.00 < x_0 < 1.25$.

$f(1.125) \approx -0.451$

Thus $1.125 < x_0 < 1.25$ and $x_0 \approx 1.2$ to the nearest 0.1.

(b) $f(1.1875) \approx f(1.19) \approx -0.125$

Thus, if $f(x_0) = 0$ then $1.19 < x_0 < 1.25$,

$f(1.22) \approx +0.036$

Thus $1.19 < x_0 < 1.22$.

Since $|f(1.22) - 0|$ is much less than $|f(1.19) - 0|$, we will try $x_0 \approx 1.21$ instead of 1.205.

$f(1.21) \approx -0.018$

Thus $1.21 < x_0 < 1.22$.

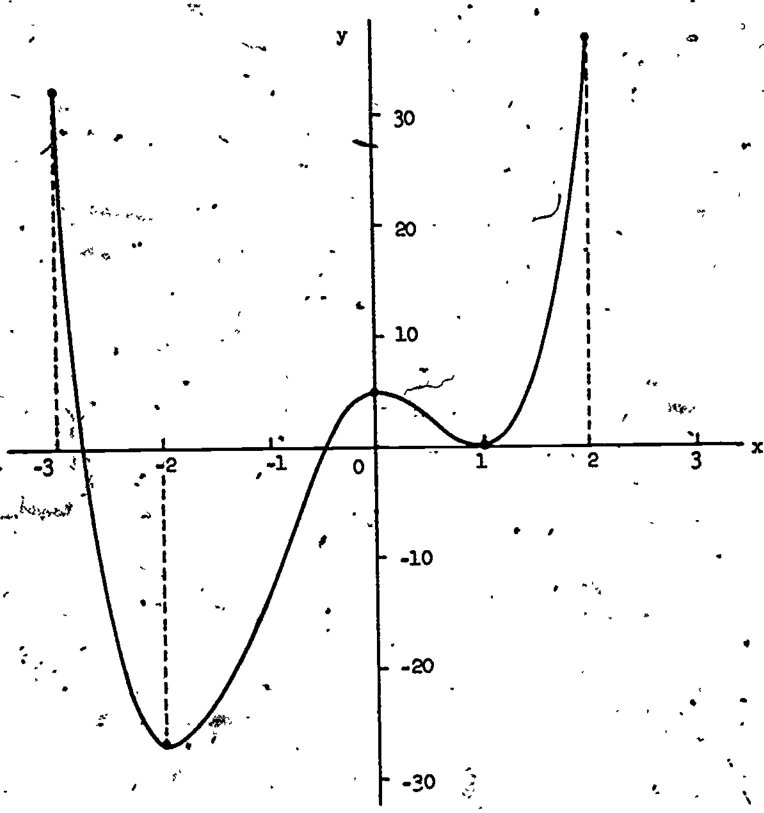
$f(1.215) \approx +0.009$

Thus $1.210 < x_0 < 1.215$ and $x_0 \approx 1.21$ to the nearest 0.01.

6. $\sqrt[3]{20} \approx 2.71$

Solutions Exercises 1-9

1.



$$y = 3x^4 + 4x^3 - 12x^2 + 5$$

We shall return to this graph in Chapter 2 when we have more equipment.

2. Your students can compare their attempts with Figure A2-2a in Appendix 2.

3. Let $g(x) = kf(x)$ which has the same zeros as $f(x)$. Then for each x_1 such that $f(x_1) = a \neq 0$ then $kf(x_1) = ka$. These two functions are not identical

   Consider another possibility. Suppose that $g(x) = p(x)f(x)$ where p has no real zeros.

   Thus f and g need not be identical.

4. (a) 1
   (b) 2
   (c) 3
   (d) 3
   (e) 2
   (f) 4
   (g) 0
   (h) 3
   (i) 0

5. f is a polynomial function.
   $g : x \to f(ax + b)$, a, b constants, $a \neq 0$.

   (a) Yes, g is a polynomial function. The degree of g is the same as the degree of f, namely n.

   (b) If $a = 1$ and $b = 0$ the graphs are the same. If $a = 1$ and $b \neq 0$ the graph of g is a translation of the graph of f. The translation if $b > 0$ being that each point of the graph of g is slid b units to the left of each corresponding point of the graph of f. If $b < 0$ then the shift of each point of f to the corresponding point of g is to the right.

   (c) If $b = 0$ then the graphs of g and f have the same y-intercepts and for corresponding highs and lows they have the same absolute ordinate values. The affect of a upon f is a scaling factor. If $0 < |a| < 1$ the graph of g is a spread out version of the graph of f. If $1 < |a|$ then the graph of g corresponds to compressing the graph of f toward the y-axis. Should we have $a < 0$, then the graph of g for $a > 0$ is folded about the y-axis. Since for $a_1 = -a_2 \neq 0$ then $f(a_1x) = f(a_2(-x))$.

   (d) The function g written in the form $g : x \to f(a(x + \frac{b}{a}))$, for general a and b, $a = 0$, implies that there is a translation of the graph of f along the x-axis, then there is a change of scale. The other form $g : x \to f(ax + b)$ implies that there is first a change of scale, then there is a translation. The actual changes of scale and translations are explained in parts (b) and (c). For polynomial functions $f(ax + b)$ and $f(a(x + \frac{b}{a}))$ are equivalent.

Teacher's Commentary

Chapter 2

THE DERIVATIVE OF A POLYNOMIAL FUNCTION

In the first six chapters our purpose is twofold: to enable the student to determine the behavior of elementary functions and to learn basic differential calculus. Accordingly, we have designed the sequence shown.

function	calculus-of-function
1	2
3	4
5	6

The heuristic development of some of the basic machinery of differential calculus is an important but secondary goal of this chapter. Our primary aim is to cultivate in the student an understanding of the behavior and possible applications of the simplest and most predictable functions of all -- polynomials. Later we shall use these concepts and techniques, and the polynomial functions themselves, as we deal with other functions and problems. We want the student to appreciate the efficiency of the fundamental concepts and techniques of calculus for studying the behavior of elementary functions. At the same time we expect that the student will begin to believe that there are some underlying principles of calculus which might be applicable to more functions than he has at his immediate disposal.

This is the first differentiation chapter. Chapter 7 is the first integration chapter. In the following paragraphs we include thoughts some of which you may wish to share with the student now, some as you begin Chapter 7, some not at all.

We want the student to know that calculus is the study of the derivative and the integral, the relationship between these concepts and their applications. The derivative and the integral may be interpreted geometrically as slope and as area, but these are only two among a wide range of interpretations and applications. We emphasize slope and area in order to introduce parts of the subject in an intuitive, geometrical way. However, the concepts of derivative and integral are universal, and their incorporation into a calculus, a system of reckoning, enables us to solve significant problems in all branches of science.

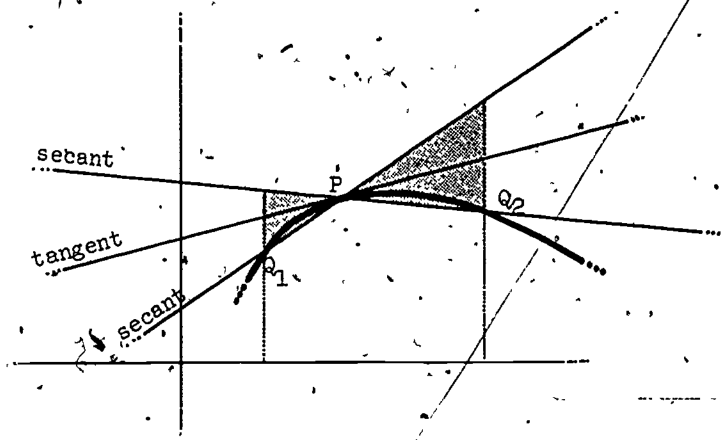
The calculus was invented to treat problems of physics. As the calculus grew into the larger branch of mathematics known as analysis its range of application expanded enormously. To analysis we owe much of the progress in the physical sciences and modern engineering, and more recently in the biological and social sciences. The concepts and operations introduced by the calculus provide the right language and the right tools for the major part of the applications of mathematics to the sciences.

The great advance which takes the calculus beyond algebra and geometry is based on the concept of limit. The basic limit procedure of the differential calculus is typified by the problem of finding the slope of a curve; the basic limit procedure of the integral calculus is typified by the problem of finding the area enclosed by a curve. The slope is found as a derivative, the area as an integral, and superficially these appear to be unrelated. But there is only one calculus: derivative and integral are complementary ideas. If we take the slope of the graph of the area function, we are brought back to the curve itself. If we take the area under the graph of the slope function, we find the original curve again. The limit concept, in its guises of derivative and integral, together with this inverse relation between the two, provides the fundamental framework for the calculus. In this chapter we deal with slope and the idea of derivative for polynomial functions.

The geometrical concept of direction for a curve can be given a precise analytical interpretation in terms of slope. The slope of a curve at a given point is a local property in the sense that it is completely determined by any arc containing the point, no matter how small. For the purpose of describing such a local property, the idea of limit is especially appropriate. The slope of a curve at a point can be defined as the limit of a set of slopes of chords (or secants of the curve. Later we discuss this limit in terms of a purely analytical concept, the derivative of a function at a point. The analytical concept, divorced from its geometrical interpretation as slope, will be seen to have other realizations. For some simple examples we show the student how to compute such limits.

It should not be thought that the methods of geometry are entirely powerless in the present context. The ancient Greeks treated the problem of defining the direction of a curve at a point by finding the tangent line at the point, the tangent being the line through the point which has the direction of the curve there. They were able to construct the tangents to all the conic sections (circle, parabola, ellipse, hyperbola) and even to such complicated curves as the spiral of Archimedes. In the end, though, the Greeks were unable to solve the problem of drawing tangents to more than a limited class of curves whose special geometrical properties made the problem tractable. What limited them was the fact that they had no general way of defining a curve, say, in terms of functions; that had to wait until the invention of analytic geometry by Descartes.

In the first two sections of this chapter we characterize the tangent to a curve at a point P as the best straight line approximation to the curve at P. Geometrically, the tangent at P is considered here to be the one straight



line through P which lies in all the wedges whose edges are secants PQ_1, where Q_1 is any point of the curve "close" to P.

The relatively intuitive work with linear approximations to polynomials is intended to give the student a feeling for what tangents to curves are, leading up to Section 3 in which the slope of a tangent is (rigorously?) defined as a limit of slopes of secants.

In the final section of this chapter, students will see that the coefficients of the terms in a polynomial function are related to its derivatives, just as their work with linear, quadratic, and cubic approximations may lead them to suspect.

Solutions Exercises 2-1

1\. (a) $y = 1 - x$

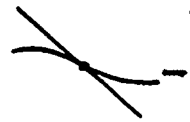
(b) $y = 4$

(c) $y = 2 + 3x$

(d) $y = 3 + 2x$

(e) $y = 1 + x$

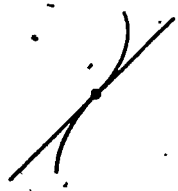
(f) $y = 1 - x$



(g) $y = 2$

(h) $y = 1 + 2x$

(i) $y = x$



(j) $y = 0$

(a) Since $-.01 < x < .01$

$$.99 < 1 + x < 1.01.$$

If $x > 0$, we have

$$.99x < (1 + x)x < 1.01x$$

and $$1 + .99x < 1 + (1 + x)x < 1 + 1.01x.$$

If $x < 0$,

$$.99x > (1 + x)x > 1.01x$$

and $$1 + .99x > f(x) > 1 + 1.01x.$$

(b) (i) For $0 < x < .01$

$1 \leq 1 + x < 1.01$

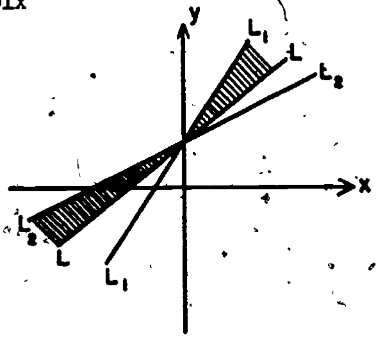
$1 + x < 1 + (1 + x)x < 1 + 1.01x$

(ii) For $-.01 < x < 0$

$.99 < 1 + x < 1$

$.99x > (1 + x)x > x$

$1 + .99x > f(x) > 1 + x.$



$L_1 : y = 1 + 1.01x$

$L : y = 1 + x$

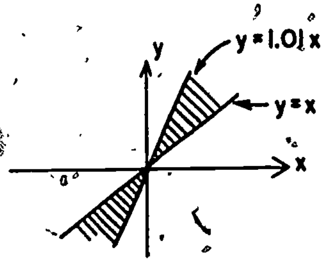
$L_2 : y = 1 + .99x$

(c) For $x \neq 0$, $x^2 > 0$. Hence $f(x) > 1 + x$, $x \neq 0$.

3. (a) (i) If $0 < x < .1$, then $0 < x^2 < .01$, $1 < 1 + x^2 < 1.01$ and $x < (1 + x^2)x = f(x) < 1.01x$.

(ii) If $0 > x > -.1$, then $0 < x^2 < .01$, $1 < 1 + x^2 < 1.01$, and $x > (1 + x^2)x = f(x) > 1.01x$.

(b) Sketch:

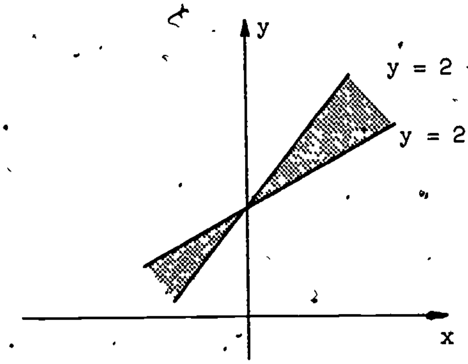


4. (a) $(0,2)$

(b) $|x| < .01 \Leftrightarrow -.01 < x < .01 \Leftrightarrow -.01 < -x < .01 \Leftrightarrow 2.99 < 3 - x < 3.01.$

If $x > 0$, (1) is equivalent to $2.99x < (3 - x)x < 3.01x$ or $2 + 2.99x < 2 + (3 - x)x = f(x) < 2 + 3.01x$, and if $x < 0$ the inequalities are reversed.

(c)



5. If $x \neq 0$, $x^2 > 0$, and hence

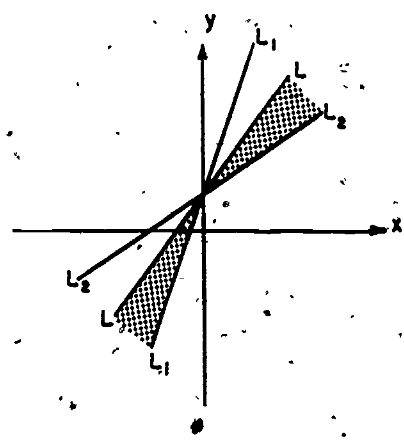
$2 + 3x - x^2 < 2 + 3x$

$L_1 : y = 2 + 3.01x$

$L : y = 2 + 3x$

$L_2 : y = 2 + 2.99x$

Refinement is shown.



6. (a) If $0 < x < .01$, then

$$-2 < -2 + x < -1.99$$

$$-2x < (-2 + x)x < 1.99x,$$

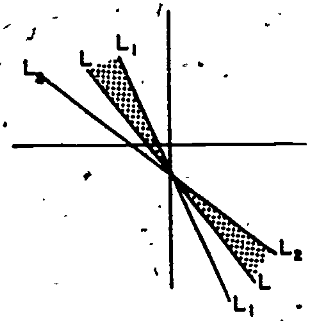
$$-1 - 2x < f(x) < -1 - 1.99x.$$

$$L : y = -1 - 2x$$

$$L_1 : -1 - 2.01x = y$$

$$L_2 : y = -1 - 1.99x$$

(b) Figure is shown.



7. (a) We can write $f : x \to 3 - 5x - 4x^2$ as

$$f : x \to 3 + (-5 - 4x)x.$$

If $|-4x| < \epsilon$, we have

$$|4x| < \epsilon \text{ or } |x| < \frac{\epsilon}{4}.$$

Thus the graph of f lies between the lines given by

$$y = 3 + (-5 + \epsilon)x$$

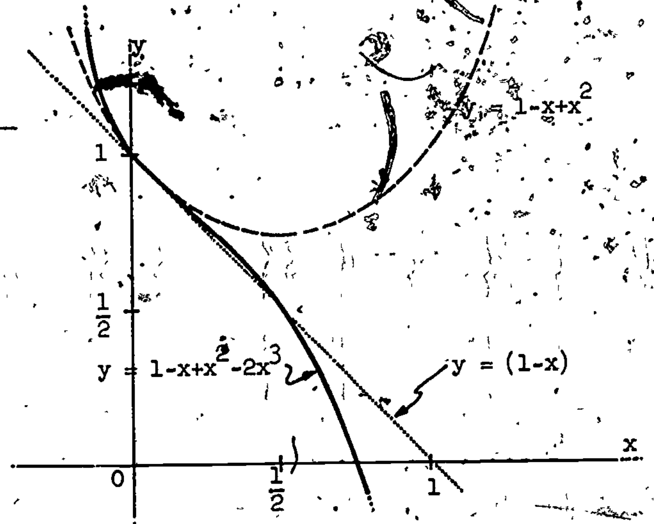
and

$$y = 3 + (-5 - \epsilon)x.$$

In particular, if $|x| < .02$, $\epsilon = .08$; and the graph of f lies between the lines given by $y = 3 - 4.92x$ and $y = 3 - 5.08x$. The slopes of these lines are, of course, -4.92 and -5.08.

(b) If $\epsilon = .002$ and $|-4x| < .002$, then $|x| < .005$, or x may assume values between $-.005$ and $.005$.

8. (a)



(b) $f(0.1) - g(0.1) = -2(0.001) = -0.002$

(c) $f(0.01) - g(0.01) = -2(0.000001) = -0.0000002$

(d) zero, because

$$\frac{f(x) - g(x)}{x^2} = \frac{(1 - x + x^2 - 2x^3) - (1 - x + x^2)}{x^2} = -\frac{2x^3}{x^2} = -2x$$

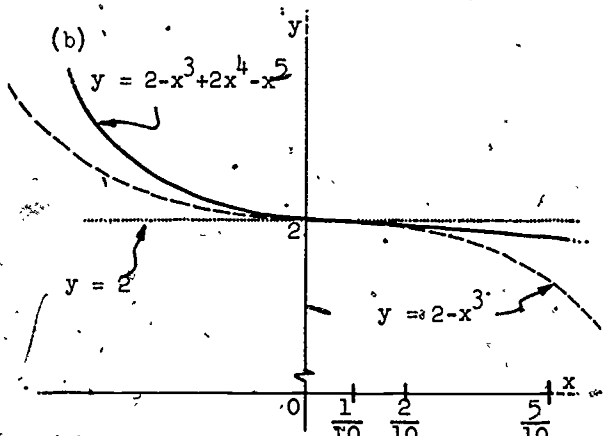
(for $x \neq 0$), and as x approaches 0, $-2x$ approaches 0 also.

9. (a) best linear approximation: $y = 2$.

best quadratic approximation: same as best linear approximation.

best cubic approximation: $y = 2 - x^3$.

(b)



The graph of f lies above the graph of its best cubic approximation g near $x = 0$ since $f(x) - g(x) = x^4(2 - x)$ which is non-negative for all $x \leq 2$.

(c) zero, because

$$\frac{f(x) - g(x)}{x^3} = \frac{(2 - x^3 + 2x^4 - x^5) - (2 - x^3)}{x^3} = \frac{2x^4 - x^5}{x^3} = 2x - x^2, \quad (x \neq 0)$$

and as x approaches 0, $2x - x^2$ approaches 0, also.

Solutions Exercises 2.2

1. (a)

2	0	-5	0	$\lfloor 2$
2	4	3	6	
2	8	19		
2	12			

$$f(x) = 2(x - 2)^3 + 12(x - 2)^2 + 19(x - 2) + 6$$

(b)

1	-7	3	4	$\lfloor 2$
1	-5	-7	-10	
1	-3	-13		
1	-1			

$$f(x) = (x - 2)^3 - (x - 2)^2 - 13(x - 2) - 10$$

(c)

3	-5	2	1	$\lfloor -1$
3	-8	10	-9	
3	-11	21		
3	-14			

$$f(x) = 3(x + 1)^3 - 14(x + 1)^2 + 21(x + 1) - 9$$

(d)

1	-2	1	-1	$\lfloor -\frac{1}{2}$
1	$-\frac{5}{2}$	$\frac{9}{4}$	$-\frac{17}{8}$	
1	-3	$\frac{15}{4}$		
1	$-\frac{7}{2}$			

$$f(x) = (x + \tfrac{1}{2})^3 - \tfrac{7}{2}(x + \tfrac{1}{2})^2 + \tfrac{15}{4}(x + \tfrac{1}{2}) - \tfrac{17}{8}$$

2. (a)

1	2	4	-3	$\lfloor 2$
1	4	12	27	
1	6	24		
1	8			

$$f(x) = (x - 2)^3 + 8(x - 2)^2 + 24(x - 2) + 27$$

$$T : y = 24(x - 2) + 27 = 24x - 21$$

(b)

$$\begin{array}{rrrr|l} 2 & 4 & 0 & 3 & \underline{-3} \\ 2 & -2 & 6 & -15 & \\ 2 & -8 & 30 & & \\ 2 & -14 & & & \end{array}$$

$$f(x) = 2(x+3)^3 - 14(x+3)^2 + 30(x+3) - 15$$

$$T : y = 30(x+3) - 15 = 30x + 75$$

(c)

$$\begin{array}{rrrr|l} 4 & -3 & 2 & 1 & \underline{-4} \\ 4 & -19 & 78 & -311 & \\ 4 & -35 & 218 & & \\ 4 & -51 & & & \end{array}$$

$$f(x) = 4(x+4)^3 - 51(x+4)^2 + 218(x+4) - 311$$

$$T : y = 218(x+4) - 311 = 218x + 561$$

(d)

$$\begin{array}{rrrrr|l} 5 & 0 & -3 & 2 & 1 & \underline{\tfrac{1}{2}} \\ 5 & \frac{5}{2} & -\frac{7}{4} & \frac{9}{8} & \frac{25}{16} & \\ 5 & 5 & \frac{3}{4} & \frac{3}{2} & & \\ 5 & \frac{15}{2} & \frac{9}{2} & & & \\ 5 & 10 & & & & \end{array}$$

$$f(x) = 5(x - \tfrac{1}{2})^4 + 10(x - \tfrac{1}{2})^3 + \tfrac{9}{2}(x - \tfrac{1}{2})^2 + \tfrac{3}{2}(x - \tfrac{1}{2}) + \tfrac{25}{16}$$

$$T : y = \tfrac{3}{2}(x - \tfrac{1}{2}) + \tfrac{25}{16} = \tfrac{3}{2}x + \tfrac{13}{16}$$

(e)

$$\begin{array}{rrrr|l} 4 & 1 & 3 & 0 & \underline{3} \\ 4 & 13 & 42 & 126 & \\ 4 & 25 & 117 & & \\ 4 & 37 & & & \end{array}$$

$$f(x) = 4(x-3)^3 + 37(x-3)^2 + 117(x-3) + 126$$

$$T : y = 117(x-3) + 126 = 117x - 225$$

(f)

$$\begin{array}{rrrr|l} 2 & 1 & -16 & -24 & \underline{-2} \\ 2 & -3 & -10 & -4 & \\ 2 & -7 & 4 & & \\ 2 & -11 & & & \end{array}$$

$$f(x) = 2(x+2)^3 - 11(x+2)^2 + 4(x+2) - 4$$

$$T : y = 4(x+2) - 4 = 4x + 4$$

3. (a)

1	-7	3	4	$\lfloor 2$
1	-5	-7	-10	
1	-3	-13		
1	-1			

T : $y = -13(x - 2) - 10 = -13x + 16$

(b)

1	-6	6	-1	$\lfloor 3$
1	-3	-3	-10	
1	0	-3		
1	3			

T : $y = -3(x - 3) - 10 = -3x - 1$

(c)

3	-4	0	0	0	$\lfloor 1$
3	-1	-1	-1	-1	
3	2	1	0		
3	5	6			

T : $y = -1$

(d)

2	-4	-5	9	$\lfloor 2$
2	0	-5	-1	
2	4	3		
2	8			

T : $g(t) = 3(t - 2) - 1 = 3t - 7$

(e)

2	-3	-12	14	$\lfloor 1$
2	-1	-13	1	
2	1	-12		
2	3			

T : $y = -12(x - 1) + 1 = -12x + 13$

(f)

2	-6	6	-1	$\lfloor 1$
2	-4	2	1	
2	-2	0		
2	0			

T : $y = 1$

4. (a) $f(x) = 2 + 9(x - 2) + 6(x - 2)^2 + 1(x - 2)^3$

Note that we could also write

$$f(x) = 2(x - 2)^0 + 9(x - 2)^1 + g(x - 2)^2 + 1(x - 2)^3.$$

(b) For $x = 1.9$ or $x = 2.1$, we have

$$.001 < .01 < .1.$$

In general for x close to 2, we have

$$(x - 2)^3 < (x - 2)^2 < x - 2.$$

(c) $y = 2$

(d) $y = 2 + 9(x - 2)$

(e) 9

(f) $y = 2 + 9(x - 2) + 6(x - 2)^2$

(g) 6

(h) From part (g) and the fact that $6 > 0$ we know that the graph is flexed (concave) upward near the point where $x = 2$.

(i) The behavior is the same.

5. (a) Since $x = a + (x - a)$ we can write

$$x^3 = a^3 + 3a^2(x - a) + 3a(x - a)^2 + (x - a)^3$$

and

$$-3x = -3a - 3(x - a).$$

Therefore,

$$x^3 - 3x = (a^3 - 3a) + (3a^2 - 3)(x - a) + 3a(x - a)^2 + (x - a)^3.$$

(b)

$$\begin{array}{cccc|l}
1 & 0 & -3 & 0 & \underline{a} \\
 & a & a^2 & a^3 - 3a & \\
\hline
1 & a & a^2 - 3 & a^3 - 3a & \\
\hline
1 & a & a^2 - 3 & & \underline{a} \\
 & a & 2a^2 & & \\
\hline
1 & 2a & 3a^2 - 3 & & \\
\hline
1 & 2a & & & \underline{a} \\
 & a & & & \\
\hline
1 & 3a & & & \\
\hline
\end{array}$$

(c) $x^3 - 3x = (a^3 - 3a) + (3a^2 - 3)(x - a) + 3a(x - a)^2 + (x - a)^3$

(d) $(x - a)^3 < (x - a)^2 < x - a$

(e) $f(a) = a^3 - 3a$

(f) $x \to (a^3 - 3a) + (3a^2 - 3)(x - a)$

(g) $3a^2 - 3$

(h) If we let $3a^2 - 3 = 0$, we have $a = \pm 1$.

(i) $(-1,2)$, $(1,-2)$

(j) $x \to (a^3 - 3a) + (3a^2 - 3)(x - a) + 3a(x - a)^2$

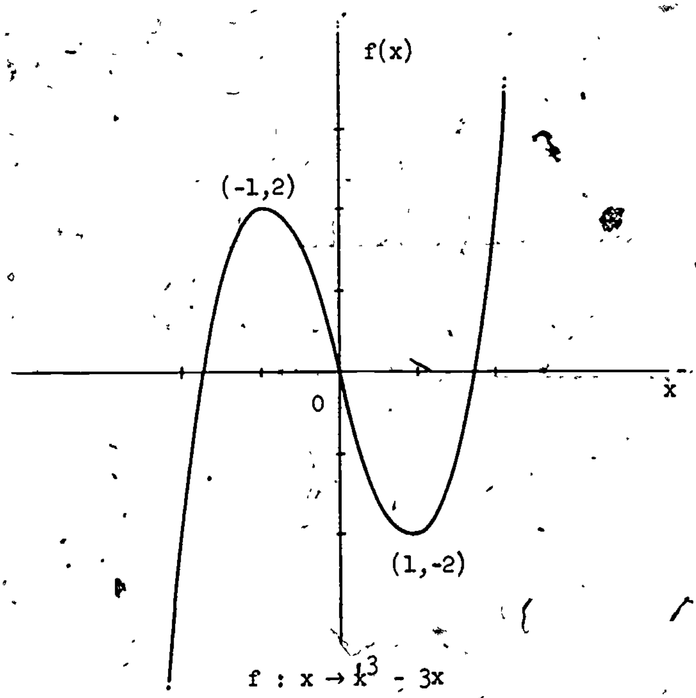
(k) $3a$

(ℓ) When $a = -1$ at $(-1,2)$, $3a < 0$; hence the graph is flexed (concave) downward. When $a = 1$ at $(1,-2)$, $3a > 0$; hence the graph is flexed (concave) upward.

(m) The point $(-1,2)$ is a relative maximum and $(1,-2)$ is a relative minimum.

(n) If we let $3a = 0$, then $a = 0$; the point (of inflection) is $(0,0)$.

(o)



$f : x \to x^3 - 3x$

2-3. The Slope as Limit of Difference Quotients

The notion of limit is introduced here and utilized without rigor. Throughout this section and most of the remainder of this text, students will be expected to deal with limits intuitively. Properties of limits are assumed without mention here and proved as theorems in Section A6-4.

Solutions Exercises 2-3

1. (a) $r(x) = \dfrac{x - a}{x - a} = 1$

   (b) $r(x) = \dfrac{x^2 - a^2}{x - a} = x + a$

   (c) $r(x) = \dfrac{x^3 - a^3}{x - a} = x^2 + ax + a^2$

   (d) $r(x) = \dfrac{x^4 - a^4}{x - a} = x^3 + ax^2 + a^2x + a^3$

2. (a) 1

   (b) $2a$

   (c) $3a^2$

   (d) $4a^3$

3. (a) 1

   (b) $2a$

   (c) $3a^2$

   (d) $4a^3$

4. (a) 1

   (b) 0

   (c) 0

   (d) 0

5. (a) $y = x$ or $y = a + (x - a)$

   (b) $y = 2ax - a^2$ or $y = a^2 + 2a(x - a)$

   (c) $y = 3a^2x - 2a^3$ or $y = a^3 + 3a^2(x - a)$

   (d) $y = 4a^3x - 3a^4$ or $y = a^4 + 4a^3(x - a)$

6. (a) $r(x) = \dfrac{mx + b - ma - b}{x - a}$

   $= m\left(\dfrac{x - a}{x - a}\right)$

   $= m, \quad x \neq a$

(b) $r(x) = \dfrac{Ax^2 + Bx + C - (Aa^2 + Ba + C)}{x - a}$

$= \dfrac{A(x^2 - a^2) + B(x - a)}{x - a}$

$= A(x + a) + B, \quad x \neq a.$

(c) $r(x) = \dfrac{A(x^3 - a^3) + B(x^2 - a^2) + C(x - a)}{x - a}$

$= A(x^2 + ax + a^2) + B(x + a) + C, \quad x \neq a.$

7. (a) m

(b) $2aA + B$

(c) $3a^2A + 2aB + C$

8. (a) m

(b) $2aA + B$

(c) $3a^2A + 2aB + C$

9. $r(x) = \dfrac{20x - 3x^2 - (20a - 3a^2)}{x - a} = 20 - 3(x + a), \quad x \neq a.$

At the point $(a, f(a))$ the slope is $20 - 6a$.

10. (a) $\lim\limits_{z \to x} \dfrac{f(z) - f(x)}{z - x} = \lim\limits_{z \to x} \dfrac{1 - z^3 - 1 + x^3}{z - x}$

$= \lim\limits_{z \to x} \dfrac{-(z^3 - x^3)}{z - x}$

$= \lim\limits_{z \to x} -(z^2 + zx + x^2)$

$= -3x^2.$

(b) $\lim\limits_{h \to 0} \dfrac{f(x + h) - f(x)}{h} = \lim\limits_{h \to 0} \dfrac{1 - (x + h)^3 - (1 - x^3)}{h}$

$= \lim\limits_{h \to 0} \dfrac{-3x^2h - 3xh^2 - h^3}{h}$

$= \lim\limits_{h \to 0} (-3x^2 - 3xh - h^2)$

$= -3x^2.$

(c) $-3x^2$

11. (a) $1 - 8a$ (c) $1 - 8x$

(b) $1 - 8a$ (d) $1 - 8x$

Remark: We use this result to begin the next section.

2-4. The Derivative

This section is designed to motivate the formulas for derivatives of the general polynomial functions given in Section 2-5. The Binomial Theorem is assumed throughout, and a reminder may be necessary before students tackle Exercise 9.

Solutions Exercises 2-4

1. (a) $$\frac{f(x+h) - f(x)}{h} = \frac{[(x+h)^2 - 1] - [x^2 - 1]}{h}$$

$$= \frac{[x^2 + 2xh + h^2 - 1] - [x^2 - 1]}{h}$$

$$= \frac{2xh + h^2}{h} = 2x + h$$

$$\therefore \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} (2x + h) = 2x$$

(b) $f'(3) = 2 \cdot 3 = 6$

(c) $$\frac{f(3+h) - f(3)}{h} = \frac{[(3+h)^2 - 1] - [3^2 - 1]}{h}$$

$$= \frac{[9 + 6h + h^2 - 1] - [9 - 1]}{h}$$

$$= \frac{6h + h^2}{h} = 6 + h$$

$$\therefore \lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = 6$$

(d)

h	0.1	0.01	0.001	-0.1	-0.01	-0.001
$\frac{f(3+h) - f(3)}{h}$	6.1	6.01	6.001	5.9	5.99	5.999

2. (a) $$\frac{f(x+h) - f(x)}{h} = \frac{[(x+h)^2 - (x+h) + 1] - [x^2 - x + 1]}{h}$$

$$= \frac{[x^2 + 2xh + h^2 - x - h + 1] - [x^2 - x + 1]}{h}$$

$$= \frac{2xh + h^2 - h}{h} = 2x - 1 + h, \quad [h \neq 0].$$

$$\therefore f'(x) = \lim_{h \to 0} (2x - 1 + h) = 2x - 1$$

(b) $$\frac{f(x + h) - f(x)}{h} = \frac{[3(x + h)^2 + 4] - [3x^2 + 4]}{h}$$

$$= \frac{[3x^2 + 6xh + h^2 + 4] - [3x^2 + 4]}{h}$$

$$= \frac{6xh + h^2}{h} = 6x + h, \quad [h \neq 0].$$

$$\therefore \quad f'(x) = \lim_{h \to 0} (6x + h) = 6x.$$

(c) $f'(x) = 4x - 1$. [Derivation similar to (a) and (b).]

3. $$\frac{f(x + h) - f(x)}{h} = \frac{[a(x + h)^2 + b(x + h) + c] - [ax^2 + bx + c]}{h}$$

$$= \frac{[ax^2 + 2axh + ah^2 + bx + bh + c] - [ax^2 + bx + c]}{h}$$

$$= \frac{2axh + ah^2 + bh}{h} = 2ax + b + ah, \quad [h \neq 0]$$

$$\therefore \quad f'(x) = \lim_{h \to 0} (2ax + b + ah) = 2ax + b.$$

4. (a) $$\frac{f(x + h) - f(x)}{h} = \frac{[(x + h)^3 + (x + h)] - [x^3 + x]}{h}$$

$$= \frac{[x^3 + 3x^2h + 3xh^2 + h^3 + x + h] - [x^3 + x]}{h}$$

$$= \frac{3x^2h + 3xh^2 + h^3 + h}{h} = 3x^2 + 1 + 3xh + h^2, \quad (h \neq 0)$$

$$\therefore \quad f'(x) = \lim_{h \to 0} (3x^2 + 1 + 3xh + h^2) = 3x^2 + 1.$$

(b) $$\frac{f(x + h) - f(x)}{h} = \frac{[(x + h)^3 - 3(x + 1)] - [x^3 - 3x]}{h}$$

$$= \frac{[x^3 + 3x^2h + 3xh^2 + h^3 - 3x - 3h] - [x^3 - 3x]}{h}$$

$$= \frac{3x^2h + 3xh^2 + h^3 - 3h}{h} = 3x^2 - 3 + 3xh + h^2, \quad (h \neq 0)$$

$$\therefore \quad f'(x) = \lim_{h \to 0} (3x^2 - 3 + 3xh + h^2) = 3x^2 - 3$$

(c) $\frac{f(x+h)-f(x)}{h} = \frac{[2(x+h)^3+(x+h)^2-6(x+h)+3]-[2x^3+x^2-6x+3]}{h}$

$= \frac{[2x^3+6x^2h+6xh^2+2h^3+x^2+2xh+h^2-6x-6h+3]-[2x^3+x^2-6x+3]}{h}$

$= \frac{6x^2h+6xh^2+2h^3+2xh+h^2-6h}{h} = 6x^2+6xh+2h^2+2x+h-6, \ (h \neq 0)$

$\therefore \ f'(x) = \lim_{h\to 0} (6x^2+6x^2h+2h^2+2x+h-6) = 6x^2+2x-6.$

5. $\frac{f(x+h)-f(x)}{h} = \frac{[a(x+h)^3+b(x+h)^2+c(x+h)+d]-[ax^3+bx^2+cx+d]}{h}$

$= \frac{[ax^3+3ax^2h+3axh^2+h^3+bx^2+2bxh+bh^2+cx+ch+d]-[ax^3+bx^2+cx+d]}{h}$

$= \frac{3ax^2h+3axh^2+h^3+2bxh+bh^2+ch}{h} = 3ax^2+3axh+h^2+2bx+bh+c, \ (h \neq 0)$

$\therefore \ f'(x) = \lim_{h\to 0} (3ax^2+3axh+h^2+2bx+bh+c) = 3ax^2+2bx+c.$

6. (a) $f'(a) = \lim_{x\to a} \frac{1+2x-x^2-1-2a+a^2}{x-a}$

$= \lim_{x\to a} (2-x-a) = 2-2a$

(b) $f'(0) = 2$

(c) $f'(\frac{1}{2}) = 1$

(d) $f'(1) = 0$

(e) $f'(-10) = 22$

7. (a) slope $= 2-2a$

(b) slope $= 2$

(c) slope $= 1$

(d) slope $= 0$

(e) slope $= 22$

8. (a) $f'(x) = 3x^2-2$ (from Exercise 5)

If $f'(x) = 4$, then $3x^2-2 = 4$

$3x^2 = 6$

$x^2 = 2$

$x = \pm\sqrt{2}$

(b) $x = \pm\sqrt{8} = \pm 2\sqrt{2}$

(c) $x = \pm\sqrt{\frac{2}{3}} = \pm\frac{\sqrt{6}}{3}$

(d) $x = \pm\sqrt{\frac{1}{3}} = \pm\frac{\sqrt{3}}{3}$

9. (a) $3x^2$

(b) $4a^3$

(c) $5x^4$

(d) $6x^5$

10. (a) slope function is the function $f': x \to 3x^2$.

(b) derivative is the function $f': x \to 4x^3$.

(c) slope at $(x,f(x))$ is $5x^4$.

(d) $f'(x) = 6x^5$.

Each of these expressions is derived in Exercise 9.

11. (a) $f(x) = (x + 1)^2 = x^2 + 2x + 1$, so $f'(x) = 2x + 2 = 2(x + 1)$ by Exercise 3.

(b) $f(x) = x(3x + 1)^2 = 9x^3 + 6x^2 + x$, so $f'(x) = 27x^2 + 12x + 1$ by Exercise 5.

(c) $f(x) = (x^2 + 2x)(3x - 1) = 3x^3 + 5x^2 - 2x$, so

$f'(x) = 9x^2 + 10x - 2$ by Exercise 5.

12. For all values a, $f'(a) = 2Aa + B = 6a - 2$; whence $A = 3$ and $B = -2$. We also have $f(1) = A(1)^2 + B(1) + C = 8$. Therefore, $3 - 2 + C = 8$; $C = 7$.

2-5. Derivatives of General Polynomial Functions

The formulas given here should come as no surprise after the work of Section 2-4. Their proofs tacitly assume that the limit of a sum is the sum of the limits.

The derivative of

$$f : x \to c_0 + c_1(x - a) + c_2(x - a)^2 + \ldots + c_n(x - a)^n$$

begins to suggest the more general property of derivatives known as the Chain Rule, i.e.,

if $$f(x) = g[h(x)]$$

then $$f'(x) = g'[h(x)] \cdot h'(x).$$

The Chain Rule will not be derived in general until Chapter 8, but specific examples will continue to crop up in connection with each of the elementary functions discussed prior to that. By Chapter 8, students may even be ready to conjecture that some such general principle is operating.

Solutions Exercises 2-5

1. (a) $f' : x \to 2x + 2$

   (b) $g' : x \to 2(x + 1)$

   (c) $f' = g'$

2. (a) (i) $f' : x \to 4x - 8$

   (ii) $g_1' : x \to 4(x + 6) - 8$

   (iii) $g_2' : x \to 4(x - 7) - 8$

   (b) (i) 4

   (ii) 4

   (iii) 4

   (c) $f'(3) = g_1'(-3) = g_2'(10)$

   (d) (i) $x \to 2(x + 2)^2 - 8(x + 2) + 9$

   (ii) $x \to 2(x + 6 - 4)^2 - 8(x + 6 - 4) + 9$

   (iii) $x \to 2(x - 7 + 9)^2 - 8(x - 7 + 9) + 9$

   In each case the same function $x \to 2x^2 + 1$ is obtained.

3. (a) (i) $F' : x \to 3x^2 - 3$

(ii) $f' : x \to 3(x - 2)^2 + 12(x - 2) + 9$

(iii) $g' : x \to 3(x + 1)^2 - 6(x + 1)$

(b) $F'(0) = f'(0) = g'(0) = -3$

(c) Each has the equation $y = -3x$.

(d) $F = f = g$

4. (a) $F' : x \to 3x^2 + 12x + 12 = 3(x + 2)^2$

(b) $f' : x \to 3(x + 2)^2$

(c) $3(x + 2)^2$

(d) 3

(e) 3

(f) 3

5. (a) $f' : x \to 10(x + 1)^9$

(b) $f(0) = 1$ and $f'(0) = 10$

(c) $y = 10x + 1$

(d) $f(-1) = 0$ and $f'(-1) = 0$

(e) $y = 0$

(The graph of f is tangent to the x-axis at the point where $x = -1$.)

(f) $f(-2) = 1$ and $f'(-2) = -10$

(g) $y = -10x - 19$ or $y = 1 + -10(x + 2)$

6. (a) $f' : x \to 15(x - 2)^{14}$

(b) $f'(1) = 15$, $f'(2) = 0$, $f'(3) = 15$

(c) $y = 2^{15} + 15 \cdot 2^{14}(x - 4)$ or $y = 32768 + 245760(x - 4)$ or $y = 245760x - 950272$

7. (a) $f' : x \to 6(x + 2)$

(b) $g' : x \to 6x + 12$

(c) $f = g$ and $f' = g'$

(d) This problem is intended to be cautionary. Without the chain rule the student must first factor or expand to obtain $F(x) = 9x^2 + 36x + 36$, whence $f' : x \to 18x + 36 \neq 2(3x + 6)$.

(e) $G' : x \to 18(x + 2)$

(f) $F = G$ and $F' = G'$

8. (a) $f' : x \to 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

(b) $f' : x \to 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \frac{x^8}{8!}$

(c) $f' : x \to x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!}$

9. (a) $f'(x) = 6x^2 - 18x - 60 = 6(x + 2)(x - 5)$, so $f'(x) = 0$ if and only if $x = -2$ or $+5$.

$\therefore$ The slope of the graph is zero at $x = -2$ and $x = +5$.

(b) The graph is horizontal.

10. (a) $f' : x \to 3x^2 - 6x$

$g' : x \to x - \frac{2}{3}$

(b) $f'(1) = -3$, $g'(1) = \frac{1}{3}$.

(c) At $(1,f(1))$ equation of tangent is $y = -3x + 2$;

at $(1,g(1))$ equation of tangent is $y = \frac{1}{3}x - \frac{4}{3}$.

(d) These tangent lines are perpendicular to one another.

11. (a) $x^7 = (x - a)^7 + 7(x - a)^6 a + 21(x - a)^5 a^2 + 35(x - a)^4 a^3 + 35(x - a)^3 a^4 + 21(x - a)^2 a^5 + 7(x - a)a^6 + a^7.$

(b) For $x \neq a$, we have

$$\frac{x^7 - a^7}{x - a} = 7a^6 + 21a^5(x - a) + 35a^4(x - a)^2 + 35a^3(x - a)^3 + 21a^2(x - a)^4 + 7a(x - a)^5 + (x - a)^6.$$

(c) $\lim_{x \to a} \frac{x^7 - a^7}{x - a} = 7a^6$

(d) Using the Binomial Theorem, we get

$$(x + \Delta x)^7 - x^7 = 7x^6\Delta x + 21x^5(\Delta x)^2 + 35x^4(\Delta x)^3 + 35x^2(\Delta x)^4 + 21x^2(\Delta x)^5 + 7x(\Delta x)^6 + (\Delta x)^7.$$

For $\Delta x \neq 0$, we have

$$\frac{(x + \Delta x)^7 - x^7}{\Delta x} = 7x^6 + 21x^5\Delta x + 35x^4(\Delta x)^2 + 35x^2(\Delta x)^3 + 21x^2(\Delta x)^4 + 7x(\Delta x)^5 + (\Delta x)^6.$$

(e) From part (d),

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} (7x^6 + 21x^5 \Delta x + 35x^4 (\Delta x)^2 + \dots + (\Delta x)^6)$$

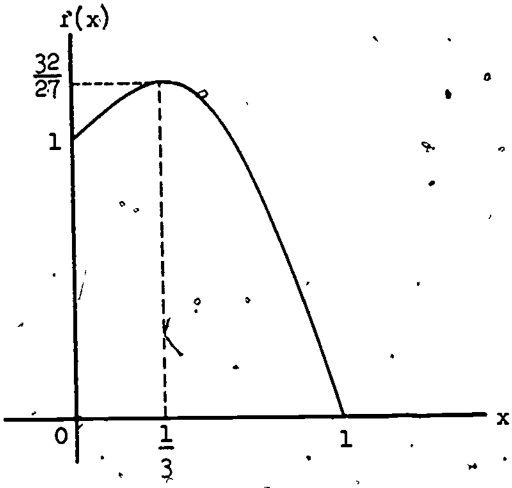
$$= 7x^6.$$

2-6. Applications of the Derivative to Graphing

In this section we define relative maxima and minima and conclude that at such points the derivative of the function involved has the value zero. The second derivative is not yet available to characterize relative maxima and minima, so the students must at this stage check the sign of $f'(x)$ on either side of the zeros of f' in order to determine whether they correspond to maxima or minima, or merely to horizontal points of inflection.

Solutions Exercises 2-6

1.



The graph corroborates the conclusions of the text.

2. (a) $f(x) = 4x^4 - 8x^2 + 1$.

$f'(x) = 16x^3 - 16x = 16x(x^2 - 1)$ is zero if $x = 0$ or $x = 1$ or $x = -1$. Since $f(-1) = f(1) = -3$ and $f(0) = 1$, we conclude that $f(0)$ is a local (relative) extremum; $f(0) = 1$ is a local (relative) maximum. Similarly, since $f(-2) = f(2) = 33$ we may conclude that $f(-1) = f(1) = -3$ are minima. The function f is decreasing throughout the intervals $x \leq -1$ and $[0,1]$; f is increasing throughout the intervals $[-1,0]$ and $x \geq 1$.

(b) $f(x) = x^4 - 4x^3$.

$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$ is zero if $x = 0$ or $x = 3$. $f(0) = 0$ and $f(3) = -27$. Since $f(-1) > f(0) > f(3)$, $f(0)$ is not a local extremum. Since $f(0) = f(4)$, $f(3)$ is an extremum. $f(3)$ is a minimum value of f. The function is decreasing throughout the interval $x \leq 3$ and increasing throughout the interval $x \geq 3$.

3. Let $x_1 > x_2 \geq 0$. Then it follows that

$$x_1 + x_2 > 0$$

and

$$x_1 - x_2 > 0.$$

Multiplying, we get

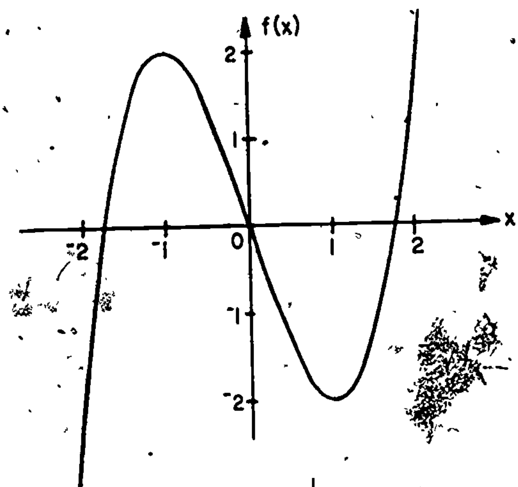
$$x_1^2 - x_2^2 > 0,$$

from which it follows that

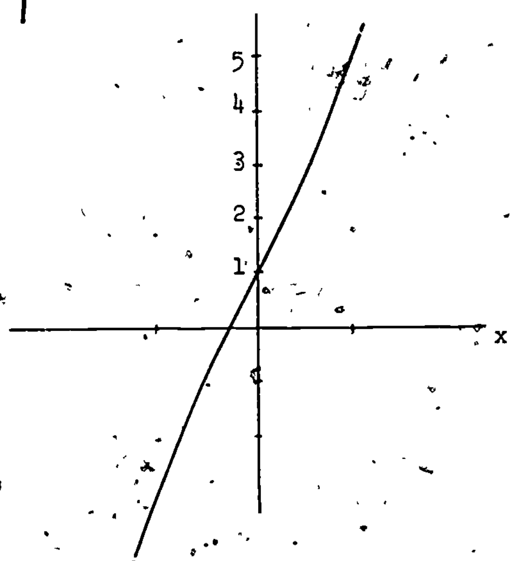
$$x_1^2 > x_2^2 .$$

We could alternatively argue that $f'(x) = 2x > 0$ for $x > 0$.

4. (a)

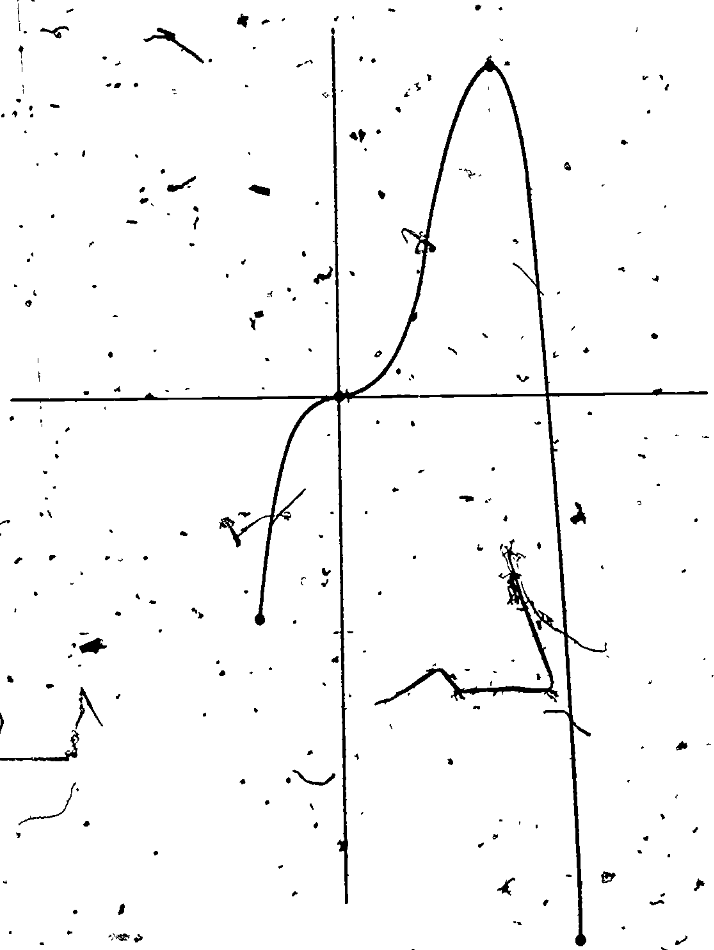


(b)



6. (a) The function f increases on $[-1,0]$, increases on $[0,2]$ and decreases on $[2,3]$.

(b)



7. The maximum value of f occurs when $x \to x^2 - 6x + 10$ is minimum; when $x = 3$, $f(3) = 8$. (Note that $x^2 - 6x + 10$ is positive (in fact $f(x) \geq 1$) for all x.)

8. The graph of a quadratic function $x \to Ax^2 + Bx + C$, $A \neq 0$, is horizontal at point a, where the slope $f'(a)$ is zero.

$$f'(a) = 2Aa + B = 0$$

So $a = \frac{-B}{2A}$ is the only point where the graph of the quadratic is horizontal.

9. (a) $f' : x \to 3Ax^2 + 2Bx + C$

(b) f' can have at most 2 zeros.

(c) From (b), we conclude that f can have at most 2 relative extrema; i.e., it can have no more than one relative maximum point and one relative minimum point.

(d) Yes. (See Complex Conjugates Theorem in Appendix 2.)

(e) Let $f'(x) = 3Ax^2 + 2Bx + C = 0$,

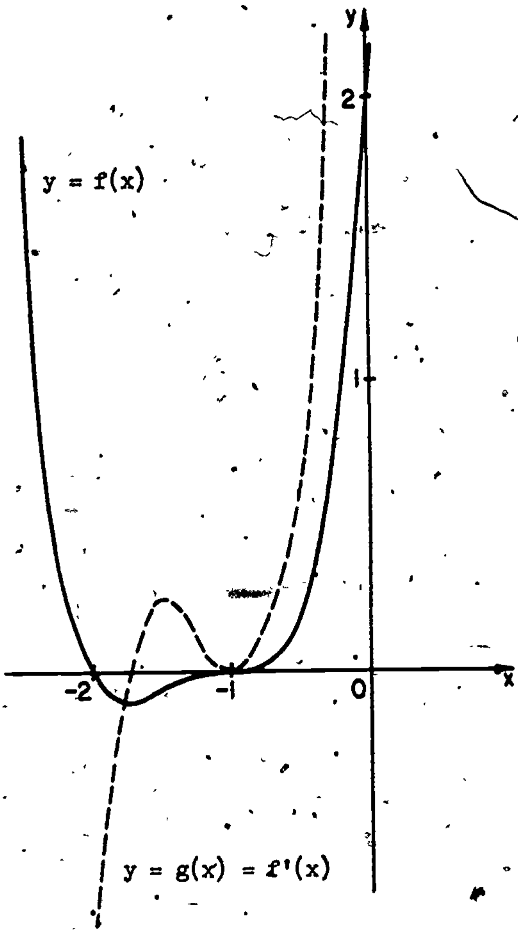
whence $x = \dfrac{-2B \pm \sqrt{4B^2 - 12AC}}{6A} = \dfrac{-B \pm \sqrt{B^2 - 3AC}}{3A}$.

Therefore

$$\frac{x_1 + x_2}{2} = -\frac{B}{3A}.$$

10. (a) $f' = g$

(b)



11. Since $f(x_1) = f(x_2) = 0$, we have,

$$Ax_1^2 + Bx_1 + C = Ax_2^2 + Bx_2 + C = 0.$$

Thus $A(x_1^2 - x_2^2) + B(x_1 - x_2) = 0$, or

$$A(x_1 + x_2) + B = 0, \quad \text{for } x_1 \neq x_2.$$

Hence, $A(x_1 + x_2) = -B$. Since $f'(a) = 2Aa + B = 0$ when $a = -\frac{B}{2A}$, we have

$$-\frac{B}{2A} = \frac{A(x_1 + x_2)}{2A} = \frac{x_1 + x_2}{2}.$$

Therefore, f has a minimum when

$$x = \frac{x_1 + x_2}{2}.$$

Alternatively we could give the following argument. We have a minimum when

$$f'(x_0) = 2Ax_0 + B = 0; \quad \text{i.e., when } x_0 = -\frac{B}{2A}.$$

Also $f(x_1) = f(x_2) = 0$, when x_1 or $x_2 = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$.

Thus $\frac{x_1 + x_2}{2} = -\frac{2B}{4A} = -\frac{B}{2A} = x_0$.

12. We have $g'(x) = 12x(x - 1)(x - 2)$. The point $(1,-1)$ is a relative maximum point. The points $(0,-4)$ and $(2,-4)$ are minimum points.

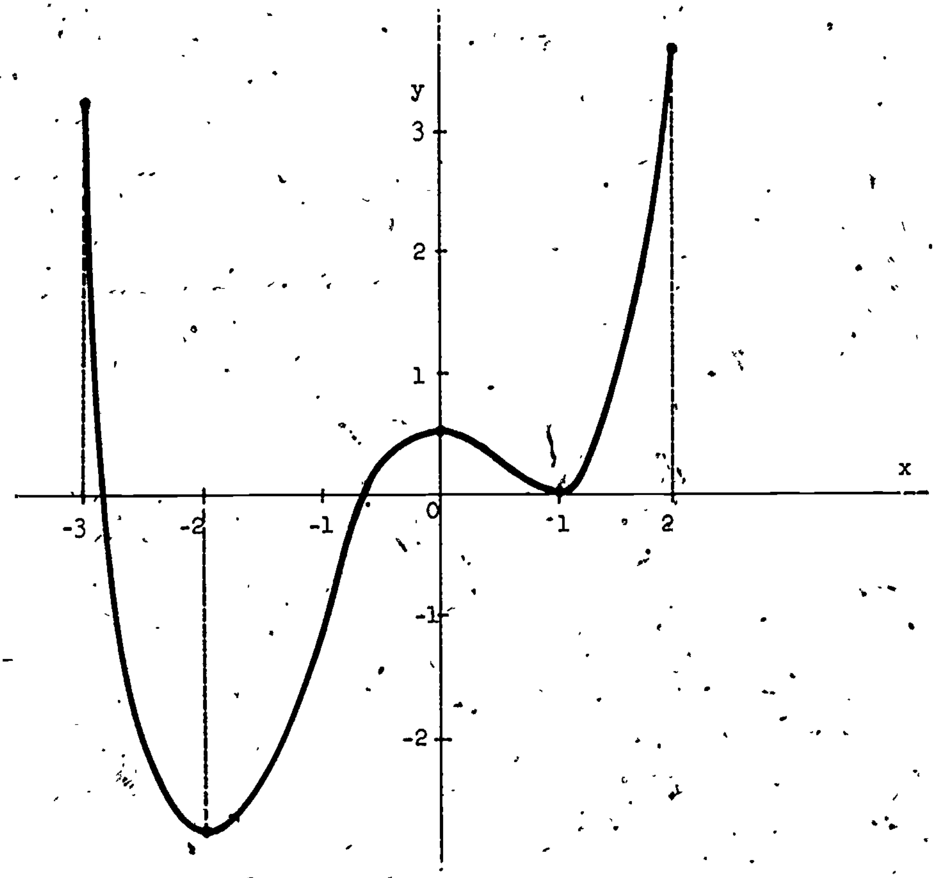
1. We have $f'(x) = 12x^3 + 12x^2 - 24x = 12x(x - 1)(x + 2)$, whence

$$f'(0) = f'(1) = f'(-2) = 0.$$

We tabulate the values of f at the zeros of the derivative and at the endpoints of an interval including the zeros.

x	-3	-2	0	1	2
f(x)	32	-27	5	0	37

The table helps us to conclude that $f(-2)$ is the minimum of $f(x)$ on the interval $[-3,2]$, $f(0)$ a maximum on $[-2,1]$, and $f(1)$ a minimum on $[0,2]$. Thus, we expect f to be a decreasing function on $-3 \leq x \leq -2$; $f(-2)$ is a local minimum. For $-2 \leq x \leq 0$ the function should be increasing and $f(0)$ is a local maximum. Over the interval $0 \leq x \leq 1$ the function should decrease again to the local minimum $f(1)$; for $x > 1$ the function should be increasing.



2. We apply the knowledge we have gained to find the (local and global) extrema of the function

$$f: x \to 4x^5 - 5x^4 - 40x^3 + 100$$

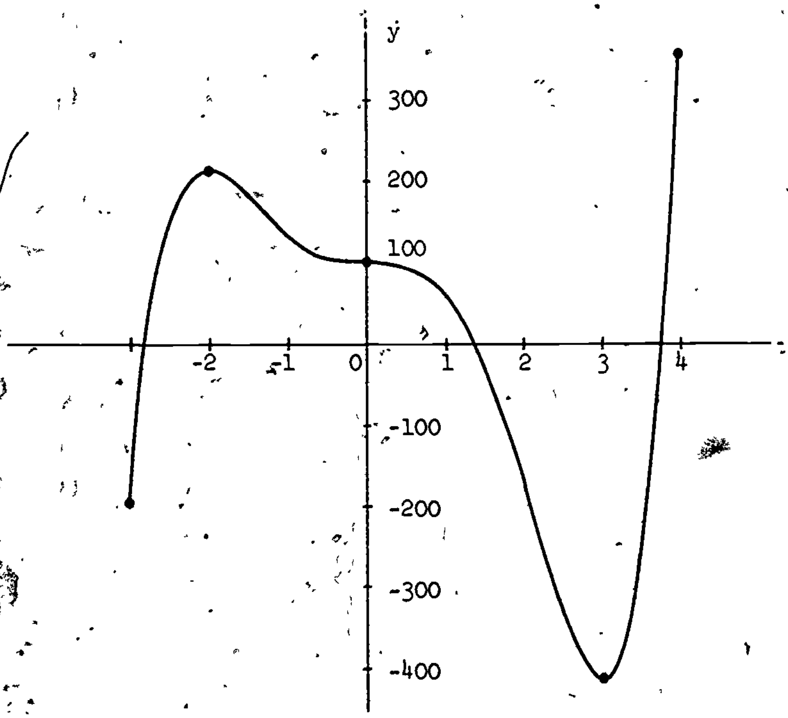
on the interval $-3 \leq x \leq 4$. We differentiate to obtain

$$f'(x) = 20x^4 - 20x^3 - 120x^2 = 20x^2(x + 2)(x - 3).$$

We tabulate the values of f at the zeros of f' and at the endpoints.

x	-3	-2	0	3	4
$f(x)$	-197	212	100	-413	356

Considering triples of consecutive values of f in this table we find that the function f increases from a (local) minimum at $x = -3$ to a (local) maximum at $x = -2$, then decreases to its (global) minimum at $x = 3$ and increases to its (global) maximum at $x = 4$. (If we were to consider the entire real axis as the domain of f then, since f' has no zeros outside the interval $[-3, 4]$ we would conclude that f is increasing for $x < -3$ and increasing for $x > 4$.) We can utilize the information of the table and a few additional plotted points to obtain an excellent idea of the behavior of the graph of f on the given interval.



3. If $f : t \rightarrow 96t - 16t^2$, then $f'(t) = 96 - 32t$.

$\therefore$ $f'(t) = 0$ if and only if $t = \frac{96}{32} = 3$.

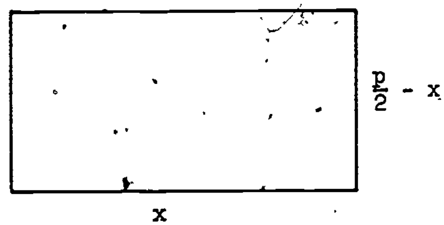
Hence, $f(t)$ is maximum when

$$t = 0, \; t = 3, \; \text{or} \; t = 6,$$

since the values of f are meaningful for the problem only on the interval $[0,6]$. Since $f(0) = 0$, $f(3) = 144$, and $f(6) = 0$, we conclude that the maximum height the ball reaches is 144 feet.

4. If x represents the length of the rectangle then $\frac{p}{2} - x$ represents the width. Hence, the area is

$$\left(\frac{p}{2} - x\right) \cdot x \quad \text{or} \quad \frac{px}{2} - x^2.$$



The function, $f : x \rightarrow \frac{px}{2} - x^2$ has the derivative $f' : x \rightarrow \frac{p}{2} - 2x$.

$$f'(x) = 0 \text{ if and only if}$$

$$\frac{p}{2} - 2x = 0$$

or

$$x = \frac{p}{4}.$$

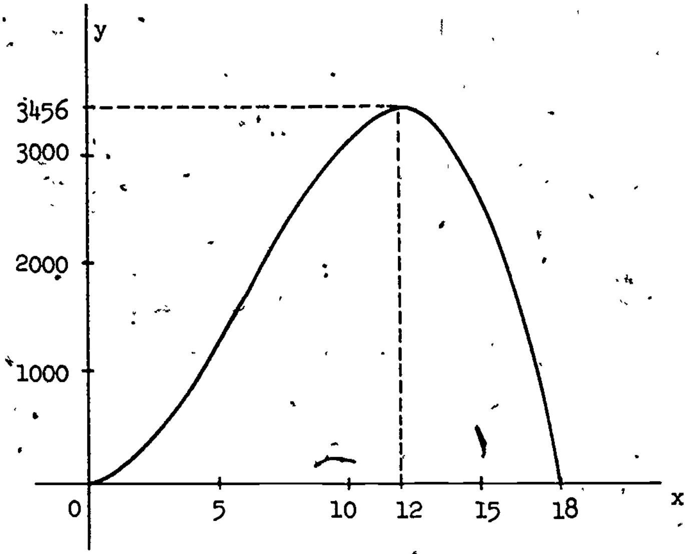
The meaningful interval for f is $[0, \frac{p}{2}]$ and on this interval f is maximum at $x = 0$, $x = \frac{p}{4}$, or $x = \frac{p}{2}$. Examination shows that, of these, $f(\frac{p}{4})$ is the greatest. Hence, the dimensions of the rectangle with largest area and perimeter p are

$$\text{length} = \frac{p}{4}$$

$$\text{width} = \frac{p}{2} - \frac{p}{4} = \frac{p}{4}.$$

Since both dimensions are the same, this maximum rectangle must be a square.

5.



We have $f'(x) = -12x^2 + 144x = 0$ when $x = 0$ or $x = 12$. On the interval $[0,18]$ the minimum value of f is $f(0) = f(18) = 0$ and the maximum value of f is $f(12) = 3456$.

6. $V = (20 - 2x)^2 x = 4(x^3 - 20x^2 + 100x) = f(x)$.
$f'(x) = 4(3x^2 - 40x + 100) = 0$, if $x = 10$ or $\frac{10}{3}$. (10 gives a minimum.) For maximum,

$$V = \frac{10}{3} \cdot \frac{40}{3} \cdot \frac{40}{3} = \frac{16,000}{27} = 592.6^+ \text{ (in cubic feet).}$$

7. Fifty feet parallel to the river; 25 feet on each side.

8. The number to be squared will be $\frac{2}{3}N$ and the other number is $\frac{N}{3}$.

9. If x is the length of the wire to be bent into the circle, the area is:

$$A = \left(\frac{24 - x}{4}\right)^2 + \frac{x^2}{4\pi} = 36 - 3x + \frac{x^2}{16} + \frac{x^2}{4\pi} = f(x)$$

$$f'(x) = -3 + \frac{x}{8} + \frac{x}{2\pi} = 0; \quad \text{if } x = \frac{24\pi}{\pi + 4} \approx 10.5^+.$$

This will give a combined area for the square and circle of 20.1^+, which is a minimum.

10. If the maximum should be required, we should have to examine the end-points (see Example 2-8d). The maximum area is 45.8^+ and corresponds to the case in which the entire wire is bent into the circle. (Properly speaking, the wire is not cut.)

11. Let s be the number of feet on the side of the square and r ft. the radius of the circle. The total area enclosed is y sq. ft., where

$$y = s^2 + \pi r^2$$

subject to the constraint that

$$4s + 2\pi r = 4,$$

so that

$$r = \frac{2(1 - s)}{\pi}.$$

We have the function

$$f: s \to y = s^2 + \frac{4}{\pi}(1 - s)^2.$$

All possible values of s are on $[0,1]$; endpoints: $(0, \frac{4}{\pi})$, $(1,1)$. Since $f'(s) = -\frac{8}{\pi}(1 - s) + 2s$, the only zero of f' is $s = \frac{4}{4 + \pi}$; $f(\frac{4}{4 + \pi}) = \frac{4}{4 + \pi}$. We observe that

$$\frac{4}{4 + \pi} < 1 < \frac{4}{\pi},$$

or

$$f(\frac{4}{4 + \pi}) < 1 < f(0).$$

(a) For minimum area the perimeter of the square should be $\frac{16}{4 + \pi}$ ft. and the circumference of the circle should be $\frac{4\pi}{4 + \pi}$ ft.

(b) For maximum area we should bend the entire wire to form a circle rather than cut it.

12. We let x feet represent the width and let y feet represent the length. Then the area is xy square feet. In order to express the area A in terms of x alone, we get y in terms of x. Since the perimeter, $2x + 2y = 72$ we have $y = 36 - x$. Thus we obtain

$$A = x(36 - x).$$

Now we can focus our attention on the behavior of the model polynomial function

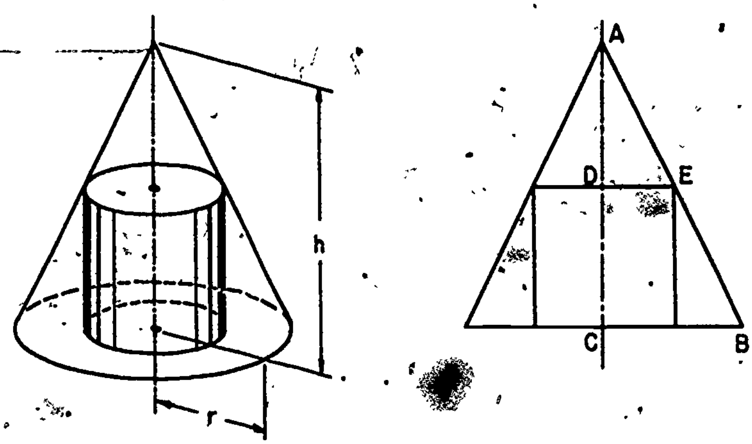
$$f : x \to 36x - x^2,$$

and its derivative

$$f' : x \to 36 - 2x.$$

Employing the same reasoning as in Example 2-8b, we examine the zeros of f'. There is only one zero, namely $x = 18$. When $x = 18$, $y = 18$ since $2x + 2y = 72$. Thus the rectangle with maximum area will be a square 18 feet on a side. We can be certain that we have obtained a maximum and not a minimum because the graph of our model function $f : x \to 36 - x^2$ is flexed downward. (Of course, we could directly apply the result of Example 2-8b.)

13.



Relating the figures we have $AC = h$ and CB [illegible]. If the height of the inscribed cylinder is y, and the radius of its circular base is x, then $DC = y$ and $DE = x$. Since triangles ADE and ACB are similar we have

$$\frac{DE}{AD} = \frac{CB}{AC}$$

or $$\frac{x}{h - y} = \frac{r}{h}, \quad \text{where } AD = h - y$$

and

(1) $$y = h - \frac{hx}{r}.$$

The volume V of the cylinder is given by

(2) $$V = \pi x^2 y = \pi x^2 (h - \frac{hx}{r})$$
$$= \pi h x^2 - \frac{\pi h}{r} x^3,$$

where h and r are constants.

The formula for the volume defines a polynomial function f which we can maximize. The derivative is

$$f' : x \to 2\pi h x - 3\pi \frac{h}{r} x^2.$$

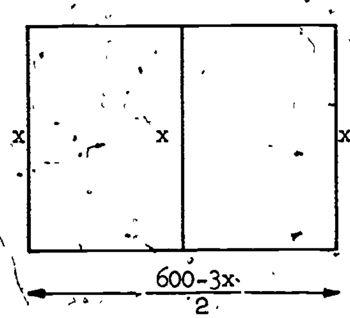
The zeros of f' are found by solving

$$0 = 2\pi h x - 3\pi \frac{h}{r} x^2$$
$$= \pi h x(2 - \frac{3}{r} x).$$

These zeros are 0 and $\frac{2}{3} r$. The cylinder will have a minimum volume when the radius of its base is 0, and a maximum volume when its radius is $\frac{2}{3} r$. To find its corresponding height we substitute $x = \frac{2}{3} r$ in (1), so that

$$y = h - \frac{2}{3} h = \frac{h}{3}.$$

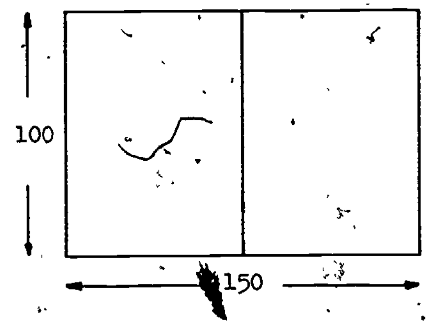
14.



$f(x) = x(\frac{600 - 3x}{2}) = 300x - \frac{3}{2}x^2$

$f'(x) = 300 - 3x$

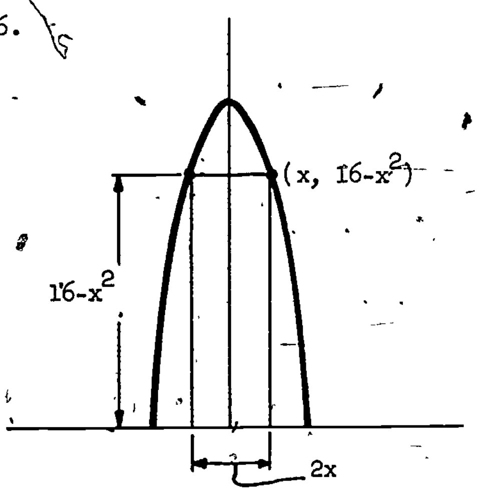
$f'(100) = 0$

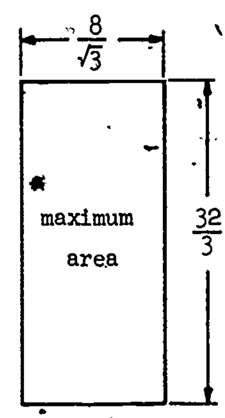


15. The dimensions will be (height × width × length) =

$(6 - 2\sqrt{3})" \times (4\sqrt{3})" \times (12 + 4\sqrt{3})"$

16.





$f(x) = 2x(16 - x^2) = 32x - 2x^3$

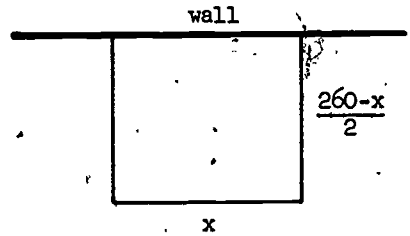
$f'(x) = 32 - 6x^2$

$f'(\frac{4}{\sqrt{3}}) = 0$

17. The area of the corral is given by

$$A = f(x) = 130x - \frac{x^2}{2},$$

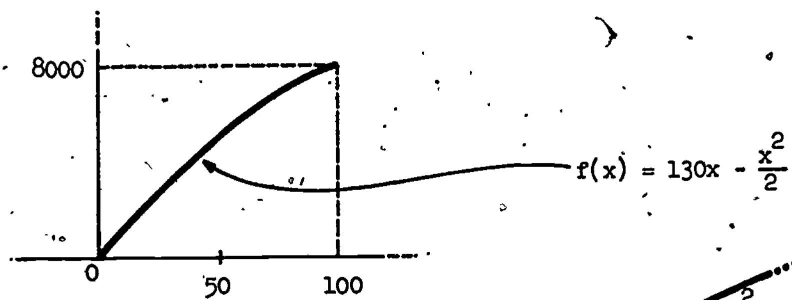
where $x \leq 100$ and $x \geq 0$.



$$f'(x) = 130 - x \text{ is zero if } x = 130.$$

$$f(0) = 0, \; f(100) = 8000.$$

Since $x \leq 100$, the extrema occur at the endpoints and the maximum area occurs at $x = 100$. The maximum area is 8000 sq. yds.



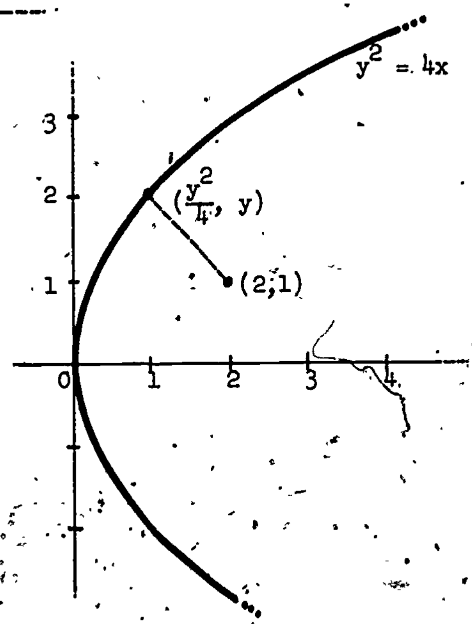
18. $D = \sqrt{\left(\frac{y^2}{4} - 2\right)^2 + (y - 1)^2}$

$K = D^2 = \frac{y^4}{16} - 2y + 5 = f(y)$

$f'(y) = \frac{y^3}{4} - 2 = 0$ if $y^3 = 8$,

$y = 2$.

The point is $(1,2)$.

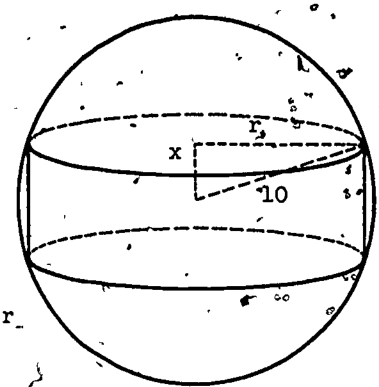


19. $V = 2\pi r^2 x = 2\pi(100 - x^2)x$

$= 200\pi x - 2\pi x^3 = f(x)$

$f'(x) = 200\pi - 6\pi x^2$

$f'(x) = 0$ when $x = \frac{10}{\sqrt{3}}$



Dimensions of inscribed right circular cylinder:

radius $= \frac{10\sqrt{6}}{3}$ inches

height $= \frac{20\sqrt{3}}{3}$ inches.

20. If we let x = the number to be found we must maximize the function

$$f : x \to x - x^2.$$

Since $f'(x) = 1 - 2x = 0$ when $x = \frac{1}{2}$, the number is $\frac{1}{2}$.

21. If x represents the length (and width) of the box in inches, $72 - x$ represents the maximum girth, so the maximum height is $\frac{72 - 3x}{2}$ or $36 - \frac{3}{2}x$. Hence, the volume $V = x^2(36 - \frac{3}{2}x) = 36x^2 - \frac{3}{2}x^3 = f(x)$

$$f'(x) = 72x - \frac{9}{2}x^2 = 9x(8 - \frac{x}{2})$$

$f'(x) = 0$ if and only if $x = 0$ or $x = 16$. Hence, dimension of parcel of this shape with maximum volume are

length = 16 inches

width = 16 inches

height = 8 inches,

and its volume is 2048 cubic inches.

22. $A = f(x) = 12x - 2x^3, \quad 0 \le x \le \sqrt{6}$

$f'(x) = 12 - 6x^2 = 0$ if $x = \sqrt{2}$

$f(0) = f(\sqrt{6}) = 0$ and $f(\sqrt{2}) = 8\sqrt{2}$.

The area is a maximum if the dimensions are $2\sqrt{2}$ by 4.

23. Let d be the depth and w the width. Then $2d + w = 14$. For the cross-sectional area A we have $A = dw$. The function to be maximized is

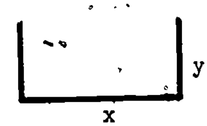
$$f : d \to 2d(7 - d) = 14d - 2d^2, \quad 0 \le d \le 7.$$

$$f'(d) = 14 - 4d = 0 \text{ if } d = \frac{7}{2}$$

$$f(0) = f(7) = 0 \text{ and } f\left(\frac{7}{2}\right) = \frac{49}{2}.$$

The trough should be $3\frac{1}{2}$ inches deep to carry the most water.

24. Volume is proportional to cross-sectional area,



$$xy = x\left(\frac{10 - x}{2}\right) = 5x - \frac{x^2}{2} = f(x)$$

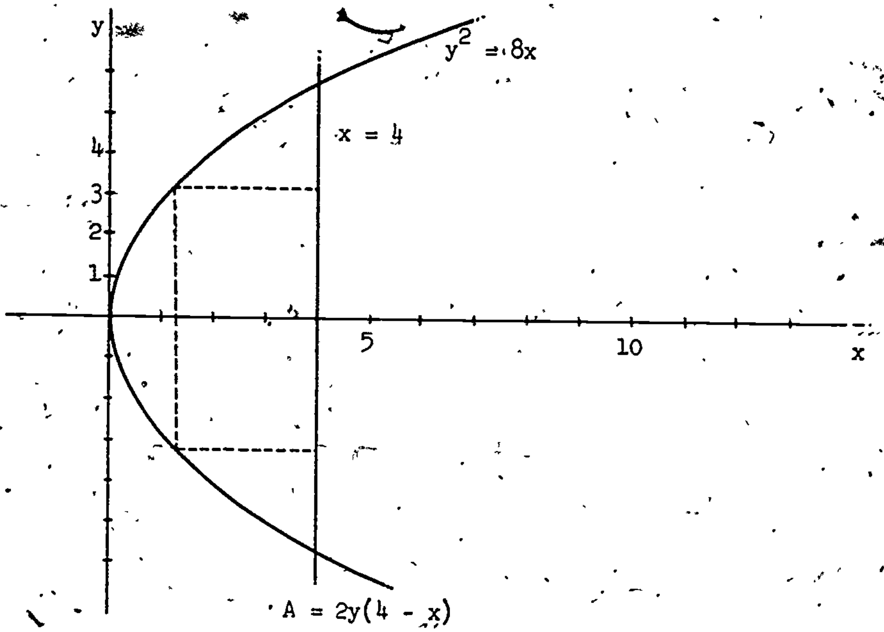
$f'(x) = 5 - x = 0$ if $x = 5$; $y = 2.5$

25. $$A = \left(\frac{p - 2x}{2}\right)x = \frac{p}{2}x - x^2 = f(x)$$

$$f'(x) = \frac{p}{2} - 2x = 0 \text{ when } x = \frac{p}{4}.$$

Since each side is $\frac{p}{4}$, the rectangle is a square.

26.



$$A = 2y(4 - x)$$

$$f(y) = 2y\left(4 - \frac{y^2}{8}\right) = 8y - \frac{y^3}{4}$$

$$f'(y) = 8 - \frac{3}{4}y^2 = 0 \text{ when } y = \sqrt{\frac{32}{3}}$$

$$A_{max} = 2\sqrt{\frac{32}{3}}\left(4 - \frac{32}{34}\right) = \frac{16}{3}\sqrt{\frac{32}{3}} = \frac{64\sqrt{6}}{9}$$

27.

$$D^2 = (1 - x)^2 + (y^2) = (1 - x)^2 + \frac{8 - x^2}{4}$$

$$= 3 - 2x + \frac{3}{4}x^2 = f(x)$$

$$f'(x) = -2 + \frac{3}{2}x = 0 \text{ when } x = \frac{4}{3}, \quad y = \frac{1}{3}\sqrt{14}$$

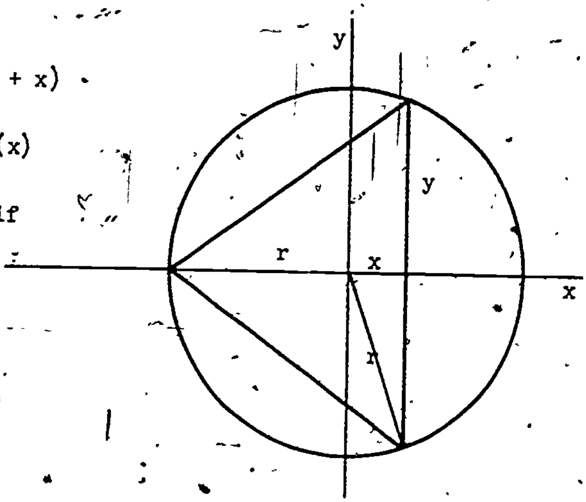
28.

$$V = \frac{\pi}{3}y^2(r + x) = \frac{\pi}{3}(r^2 - x^2)(r + x)$$

$$= \frac{\pi}{3}(r^3 + r^2x - rx^2 - x^3) = f(x)$$

$$f'(x) = \frac{\pi}{3}(r^2 - 2rx - 3x^2) = 0 \text{ if}$$

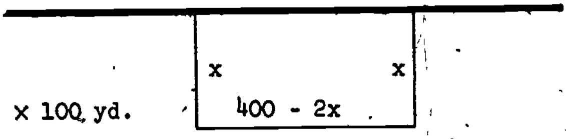
$$x = \frac{r}{3}, \; h = r + x = \frac{4}{3}r.$$



29. $A = x(400 - 2x) = f(x)$

$f'(x) = 400 - 4x$

If $x = 100$, pasture is 200 yd. × 100 yd.



30. (a) Area $= x(120 - 3x)$

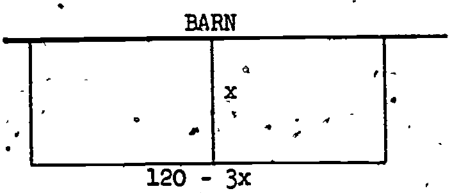
$f(x) = 120x - 3x^2$

$f'(x) = 120 - 6x.$

If $f'(x) = 0.$ $x = 20,$

$120 - 3x = 60$

Area is 1200 square feet



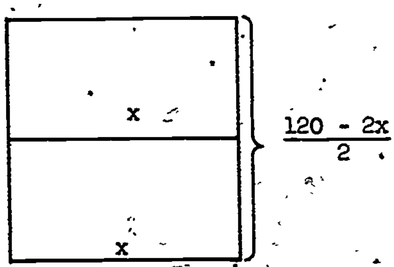
(b) Area $= x(\frac{120 - 2x}{2})$

$= 60x - x^2 = f(x)$

$f'(x) = 60 - 2x.$

If $f'(x) = 0$, $x = 30$, $\frac{120 - 2x}{2} = 30.$

Area is 900 square feet.



31. $G\ I = N[1.50 - (N - 10).03] = 1.80N - .03N^2$ where N = number of thousands.

$$1.80 - .06N = 0, \quad \text{if } N = 30.$$

30,000 labels will produce maximum gross income for printer.

32. If x represents the number of weeks and P the profit, then

$$P = (100 + 20x)(5 - \frac{x}{4}) = 500 + 75x - 5x^2 = f(x),$$

$$f'(x) = 75 - 10x$$

$$f'(x) = 0, \quad x = 7\frac{1}{2}.$$

The answer is seven weeks; the eighth week will not add to his profit. (The number $7\frac{1}{2}$ is not an answer, since it is not in our domain.)

33. If x represents the number of additional tree plantings, then the crop

$$C = (30 + x)(400 - 10x) = 12{,}000 + 100x - 10x^2 = f(x).$$

$$f'(x) = 100 - 20x = 0 \text{ if } x = 5.$$

Total number of trees per acre is 35.

34. Profit $= (3x + 6)(2 - \frac{x}{3}) = -x^2 + 4x + 12 = f(x)$, where x represents the number of weeks he should wait. $f'(x) = -2x + 4 = 0$ when $x = 2$. He should ship in 2 weeks, or July 15.

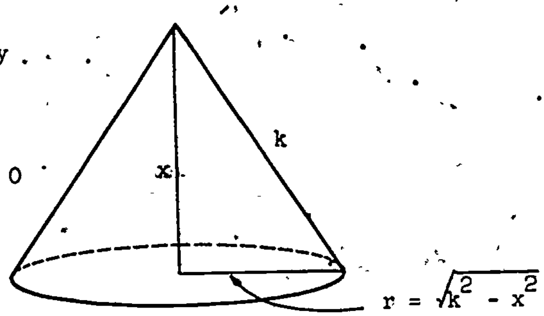
35. If we take x as the number of dollars added to the rent, the profit $= (80 - \frac{x}{2})(54 + x) = 4320 + 53x - \frac{x^2}{2} = f(x)$, and $f'(x) = 53 - x = 0$ when $x = 53$. Since $\frac{x}{2}$ must be an integer, we use either 52 or 54 as rent increases, so that the rent is either \$112 or \$114 per month. (As was the case in Number 26, the answer obtained is not in our domain, so the nearest members of the domain have to be checked.)

36. The volume of the cone is given by

$V = \frac{\pi x}{3}(k^2 - x^2) = f(x)$, where

$0 < x < k$. $f'(x) = \frac{\pi}{3}(k^2 - 3x^2) = 0$

if $x = \frac{k}{\sqrt{3}}$.

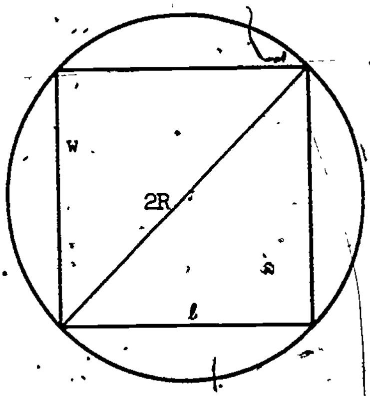


$f(\frac{k}{\sqrt{3}}) = \frac{2\sqrt{3}\,\pi k^3}{27}$, $f(0) = 0$, and

$f(k) = 0$. Thus, the maximum volume is at the interior point $x = \frac{k}{\sqrt{3}}$.

The maximum volume is $\frac{2\sqrt{3}\,\pi k^3}{27}$ cubic units.

37. We express the area of the rectangle in terms of the length of one side.



$$4R^2 = \ell^2 + w^2; \quad \ell = \sqrt{4R^2 - w^2}$$

$$A = \ell w = (\sqrt{4R^2 - w^2}) \cdot w = F(w).$$

We want to maximize the function

$$F : w \to w\sqrt{4R^2 - w^2} \quad \text{on} \quad [0,2R].$$

For the maximum $F(w)$ we shall determine the value of w for which

$$f : w \to w^2(4R^2 - w^2) = 4R^2w^2 - w^4$$

is maximum. Accordingly we seek an appropriate zero of

$$f' : w \to 8R^2w - 4w^3 \quad \text{on} \quad [0,2R].$$

Taking into account the constraints of the problem we conclude that of the zeros of f'

$$w = \sqrt{2}\, R$$

is the one which maximizes f and F. Thus, the rectangle of maximum area is a square of side $\sqrt{2}\, R$.

38. By the figure and arguments of Number 37 we see that we need to maximize the function

$$G : w \to 2\sqrt{4R^2 - w^2} + 2w \quad \text{on} \quad [0,2R].$$

Equivalently, we must maximize the function

$$g : w \to 4(4R^2 - w^2) + 8w\sqrt{4R^2 - w^2} \quad \text{on} \quad [0,2R].$$

Simplifying, we get

$$g(w) = 16R^2 + 8w\sqrt{4R^2 - w^2}, \quad 0 \leq w \leq 2R.$$

2-8. Rate of Change: Velocity and Acceleration

Velocity and acceleration are introduced as examples of the derivative interpreted as a rate of change. It will probably be obvious that acceleration is a "second" derivative of the original position function, but this terminology should be avoided until the next section. Whenever notation problems arise we have renamed the velocity function so that we may concentrate here on the notion of rates of change without introducing f''.

Solutions Exercises 2-8

1. (a) Rate of change of area of a circle with respect to its radius r is the derivative of $r \to \pi r^2$, which is $r \to 2\pi r = C$.

   (b) From Example 2-9c the rate of change of the volume is the derivative of $r \to \frac{4}{3}\pi r^3$, which is $r \to 4\pi r^2 = S$.

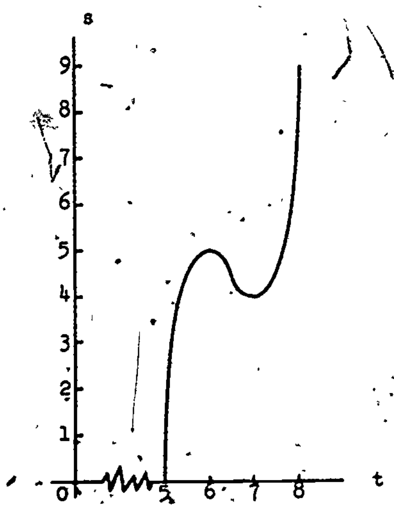
2. (a) By definition the velocity function is the derivative of the distance function. We know that the polynomial function

$$f : t \to s = 2t^3 - 39t^2 + 252t - 535$$

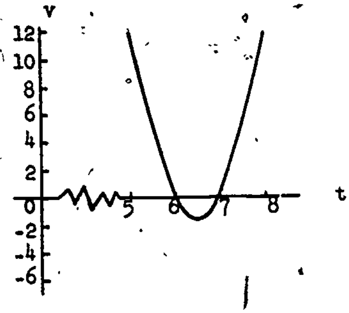
   has for its derivative the function

$$f' : t \to v = 6t^2 - 78t + 252.$$

   (b)



(c)



(d) The zeros are 6 and 7.

(e) The velocity is zero at $t = 6$ and $t = 7$.

(f) When $t = 6$.

(g) Since $s = 5$ when $t = 6$, the greatest distance is 5 units.

(h) Since, from part (g), the greatest distance on $[5,7]$ is 5 units, we require that

$$2t^3 - 39t^2 + 252t - 535 = 5.$$

We must solve the equation

$$2t^3 - 39t^2 + 252t - 540 = 0.$$

Since, by part (f), we know that $t = 6$ is a root, we can factor to obtain

$$(t - 6)(2t^2 - 27t + 90) = 0$$

or

$$(t - 6)^2(2t - 15) = 0.$$

Therefore, $s = 5$ again when $t = 7\frac{1}{2}$.

(i) At the endpoint when $t = 8$.

(j) The speed (absolute value of velocity) is greatest on $[6,7]$ when $t = 6\frac{1}{2}$.

3. $12t - 78$ units distance per units time per units time.

4. (a) $f : t \to s = 88(t - \frac{t^2}{10})$

$f' : t \to v = 88(1 - \frac{t}{5})$

when $v = 0$, $t = 5$; hence 5 seconds are required.

(b) When $t = 5$, $s = f(5) = 220$. Therefore, the car will go 220 feet (alas, sometimes less).

(c) $\alpha \approx -38.7$ (ft./sec.^{2})

(d) Distance = 25 ft.

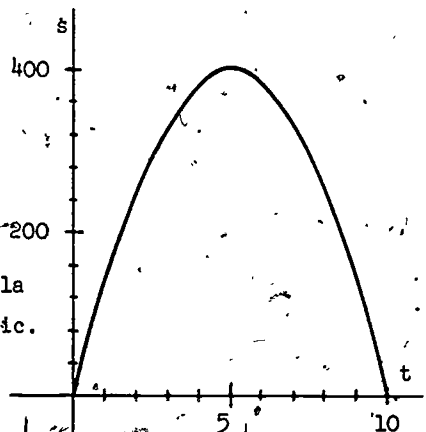
5. (a) For 8 seconds, $s = 256$ feet. For 9 seconds, $s = 144$ feet. At $t = -1$, or $t = 11$, s is negative and may be interpreted as below the surface of the ground. The maximum height seems to be 400 ft. reached after 5 seconds.

(b) A table as shown where there is a unique number for s under each t represents a function. The domain is $0 \leq t \leq 10$ or the interval $[0,10]$. The range is $0 \leq s \leq 400$ or the interval $[0,400]$.

(c) (1) Yes

(2) No

(d) Every real number for t over the interval $0 \leq t \leq 10$, would produce a value for s, so we were justified in connecting the points. The fact that the graph appears to be a parabola suggests that the function is quadratic.



(e) $s = f(t) = At^2 + Bt + C$

For $t = 0$ $\quad f(0) = C = 0$

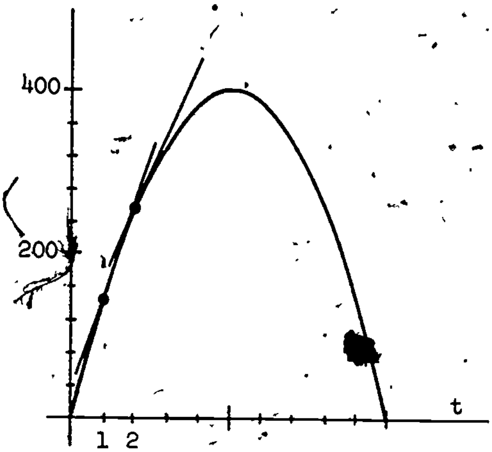
$t = 1$ $\quad f(1) = A + B = 144$

$t = 2$ $\quad f(2) = 4A + 2B = 256$

$A = -16$, $B = 160$

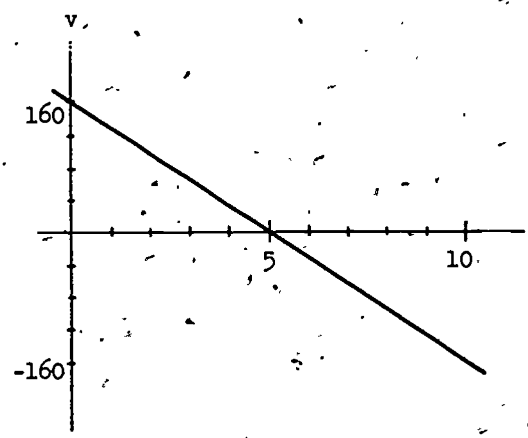
$A = -16$, $B = 160$, and $C = 0$

(f) Slope of chord through points (1, 144) and (2, 256) = 112. At $t = 1$, it would be a little larger (steeper) than 112; at $t = 2$, slightly smaller than 112.



(g) Slope is measured in units of feet per second. Velocity is the usual name. Positive values would indicate upward movement; zero, a resting position; while negative values would indicate a falling or downward movement.

(h) At $t = 1$, $v = 128$ and at $t = 2$, $v = 96$.



(i) 112 ft./sec. for each.

(j) The slope of the line $v = 160 - 32t$ is in units of feet per second per second. The word in physics is acceleration. -32 refers (in feet per second per second (ft./sec.^{2})) to the acceleration due to the force of gravity at sea level.

6. (a) When $s = 0$, $16t(10 - t) = 0$, $t = 0$ or $t = 10$, therefore, the projectile strikes the ground after 10 seconds.

(b) Since $f : t \to s = 160t - 16t^2$ and $f' : t \to v = 160 - 32t$, the velocity after t seconds is $160 - 32t$ ft./sec.

(c) When $t = 0$, $v = 160$; hence the initial velocity is 160 ft./sec.

(d) From part (a) we know that the projectile strikes the gound after 10 seconds. When $t = 10$, $v = -160$. Therefore, the impact velocity is -160 ft./sec.; i.e., the pellet strikes the ground with a speed of 160 ft./sec.

(e) 384 ft.

(f) 384 ft.

(g) Since $f'(5) = 0$, the projectile reaches its maximum height after 5 seconds.

(h) Since $f(5) = 400$, the projectile reaches a maximum height of 400 ft.

(i) From parts (f) and (h) we compute

$$400 + (400 - 384)$$

to obtain a total distance of 416 feet after 6 seconds.

7. (a) parabola

(b) $f' : t \to v = 96 - 32t$

(c) When $t = 1$, $s = 96 - 16 = 80$; hence, the ball is 80 ft. above the ground.

(d) When $t = 5$, $s = f(5) = 80$; hence, the ball is 80 feet above the ground.

(e) Since the ball reaches a maximum height of 144 feet, we compute

$$144 + (144 - 80)$$

to obtain a total distance of 208 feet after 5 seconds.

(f) Since $v = 96$ when $t = 0$, the initial velocity is 96 ft./sec.

(g) Since $s = 0$ when $t = 0$ or $t = 6$, the ball is in the air for 6 seconds.

(h) Since $v = -96$ when $t = 6$, the impact velocity is -96 ft./sec; i.e., the ball strikes the ground at a speed of 96 ft./sec.

(i) It is -32 ft./sec.^{2}

(j) $g : t \to s = -16t^2 + 96t + 200$

(k) $G : t \to s = 16t^2 + 96t + 200$

(ℓ) $F : t \to 16t^2$

8. $f : t \to ct^2$

$f'(t) = 2ct$

$= kt$, where $k = 2c$ is a positive constant.

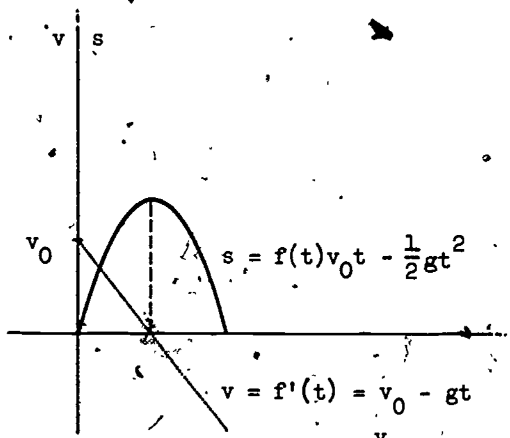
9. (a) $$f(t) = v_0 t - \frac{1}{2} gt^2.$$

The velocity is given by

$$f'(t) = v_0 - gt,$$

whence $$f'(t_0) = v_0 - gt_0.$$

(b)



(c) $f'(t) = v_0 - gt = 0$ if $t = \frac{v_0}{g}$.

(d) $f(t) = t(v_0 - \frac{1}{2} gt) = 0$ if $t = 0$ or $t = \frac{2v_0}{g}$.

The velocity on return to the initial 20 foot level is given by

$$f'(\frac{2v_0}{g}) = -v_0.$$

(e) Let v_i ft/sec = the impact velocity.

We know that $(v_i)^2 - (v_0)^2 = 2gs$ since $(v_i)^2 = (v_0 + gt)^2$.

Since the object is traveling downward, g is positive. Thus

$(v_i)^2 = (v_0)^2 + 2gs$ and $(v_i)^2 = (v_0)^2 + 60g$ since $s = 30$.

Hence, $$v_i = \sqrt{(v_0)^2 + 60g}.$$

10. From Exercise 9, the height of the ball at time t is given by the function

$$f: t \to V_0 t - 16t^2.$$

This function is maximum when $f'(t) = 0$.

$$f'(t) = V_0 - 32t = 0$$

or $$t = \frac{V_0}{32}.$$

The maximum height in feet is, therefore,

$$f\left(\frac{V_0}{32}\right) = V_0\left(\frac{V_0}{32}\right) - 16\left(\frac{V_0}{32}\right)^2 = \frac{V_0^2}{32} - \frac{V_0^2}{64} = \frac{V_0^2}{64}.$$

11. From Exercise 10, the maximum height in feet of the ball from the point of release is

$$\frac{V_0^2}{64}.$$

Hence, from ground level the maximum height in feet is

$$\frac{V_0^2}{64} + 4.$$

$$\frac{V_0^2}{64} + 4 = 148$$

$$V_0 = \pm 96.$$

Since the toss must be in the positive (up) direction, Elsie must throw the ball 96 ft./sec.

12. The height above the ground of the dropped ball is $128 - 16t^2$. The height of the tossed ball is $64t - 16t^2$. The balls collide when

$$128 - 16t^2 = 64t - 16t^2$$

i.e., when $t = 2$. The velocity of the dropped ball is given by the derivative of $128 - 16t^2$, i.e., $-32t$.

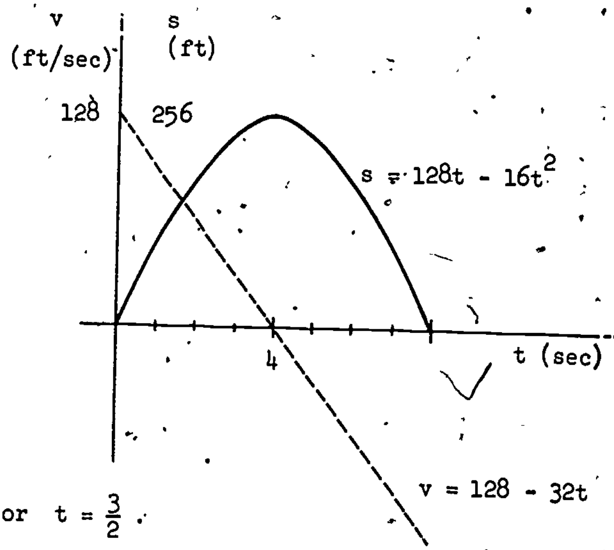
At $t = 2$, its velocity is -64 ft./sec. The velocity of the tossed ball is given by the derivative of $64t - 16t^2$, i.e., $64 - 32t$.

At $t = 2$, its velocity is 0.

13. $60\,t_1 = 30\,t_2$, $V_{av} = \dfrac{60t_1 + 30t_2}{t_1 + t_2} = \dfrac{120t_1}{3t_1} = 40(\text{mph})$

14. (a) $4 < t < 8$

(b) $v = 0$ and $t = 4$



15. (a) When $s = 12$, $t = \frac{1}{2}$ or $t = \frac{3}{2}$.

When $t = \frac{1}{2}$, $v = 16$; when $t = \frac{3}{2}$, $v = -16$.

The velocity is 16 ft./sec. when the ball first reaches a height of 12 ft, and -16 ft./sec. when it again reaches a height of 12 ft. (The speed of the ball is the same each time it is at a height of 12 ft.)

(b) $v = 0 = 32 - 32t$, at $t = 1$.

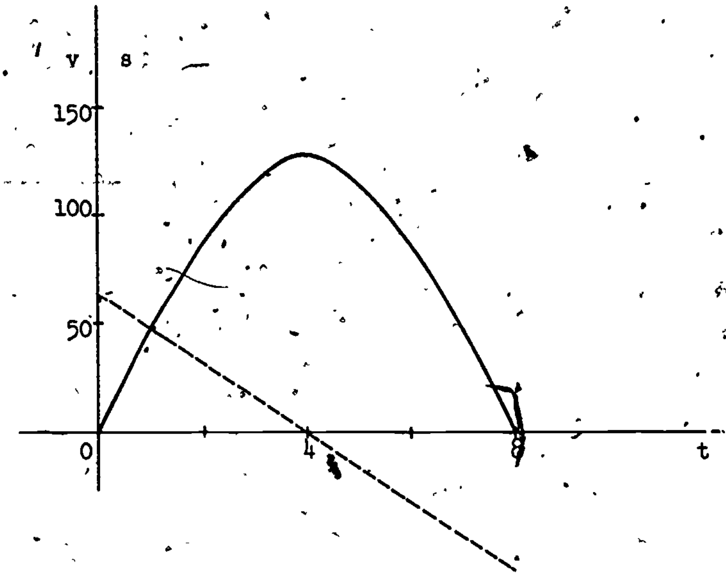
<u>One</u> second after being thrown it reaches its highest point, <u>16 feet</u>.

16. (a) $s = 64t - 8t^2$ and maximum $s = 128$ since $m = 64 - 16t$ and $t = 4$ for maximum s. Since $t_n \not< 4$, $\bar{s} = 8(t - 4)^2 = 8t^2 - 64t + 128$. But $\bar{s}$ is the distance from 128 back toward zero and so we may write:

$$s = 128 - \bar{s} = 64t - 8t^2 \quad \text{for} \quad t \geq 4.$$

Since the upward travel is described by $s = 64t - 8t^2$ for $0 \leq t \leq 4$, we have for the entire trip the function $t \to s = 64t - 8t^2$, $0 \leq t \leq 8$ since s again equals zero when $t = 8$.

(b)



1. $f'': t \to 12t - 78$

2. (a) The point $(0,0)$ is a minimum.

   (b) The point $(0,0)$ is a point of inflection.

3. (a) $f'(x) = 20x^4 + 20x^3 - 60x^2 - 100x - 40$

   $= 20(x + 1)^3(x - 2)$

   $f''(x) = 80x^3 + 60x^2 - 120x - 100$

   $= 20(x + 1)^2(4x - 5)$

   (b) The zeros of f' occur at $x = -1$ and $x = 2$. We attempt to apply the second derivative test and obtain $f''(-1) = 0$, $f''(2) = 540$. Since $f''(2) > 0$ it follows that $f(2)$ is a ~~local~~ minimum. The criterion of this section gives us no information about $f(-1)$ (but we observe that there is a reversal of a sign of f' from positive to negative so that $f(-1)$ is a local maximum).

4. (a) $f'(x) = x^4 - 2x^2 + 1 = (x^2 - 1)^2$

   (b) $f''(x) = 4x^3 - 4x = 4x(x + 1)(x - 1)$

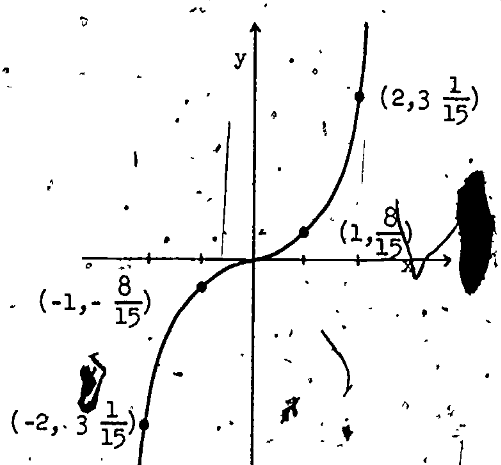
   (c) $f'(-1) = 0$

   (d) $f''(-1) = 0$

   (e) $f'(x) \geq 0$ for all x, so the graph of f is rising (or horizontal) for all x. The graph of f is horizontal at $x = \pm 1$.

   $f''(x) < 0$ for all $x < -1$, $\therefore$ concave
   $f''(x) > 0$ for $-1 < x < 0$, $\therefore$ convex
   $f''(x) < 0$ for $0 < x < 1$, $\therefore$ concave
   $f''(x) > 0$ for all $x > 1$, $\therefore$ convex.

   (f) graph:



5. The relative maximum point is $(-2,0)$.

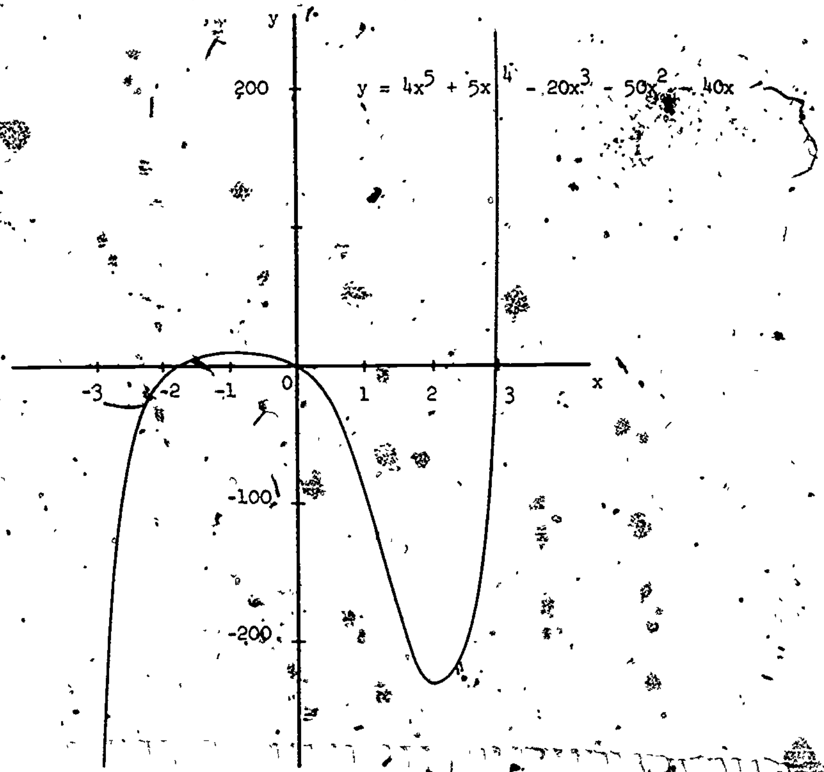
The relative minimum point is $\left(\frac{2}{3}, -9\frac{13}{27}\right)$.

6. $f(x) = 4x^5 + 5x^4 - 20x^3 - 50x^2 - 40x$

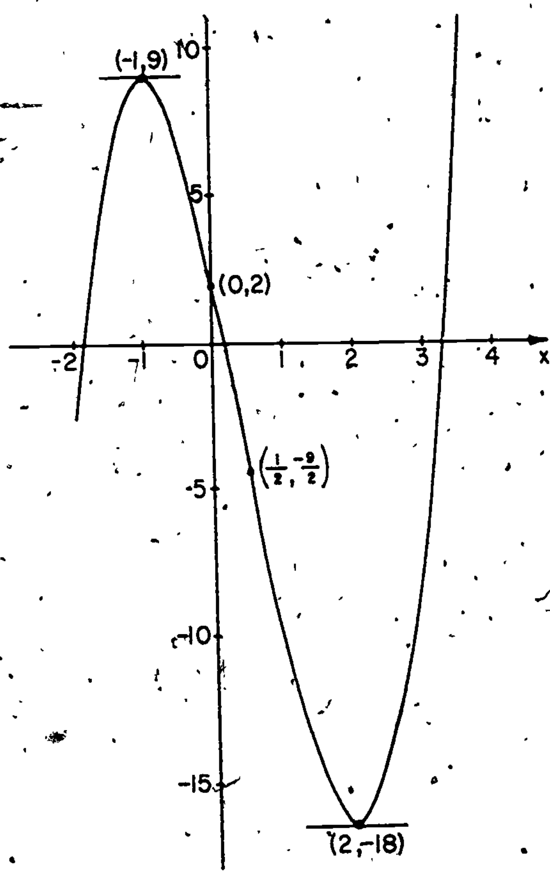
$f'(x) = 20(x + 1)^3(x - 2)$

$f''(x) = 20(x + 1)^2(4x - 5)$

The graph is flexed downward for $x < \frac{5}{4}$ and flexed upward for $x > \frac{5}{4}$; $f(-1)$ is a local (relative) maximum, and $f(2)$ is a local (relative) minimum. $f(-1) = 11$, $f(0) = 0$, $f(\frac{5}{4}) = -\frac{454}{5}$, $f(2) = -232$.



7.



8. (a) (1,1) is a point of inflection (iii).

(b) (1,1) is a relative minimum point (ii).

(c) (1,1) is a relative maximum point (i).

(d) (1,1) is none of these (iv).

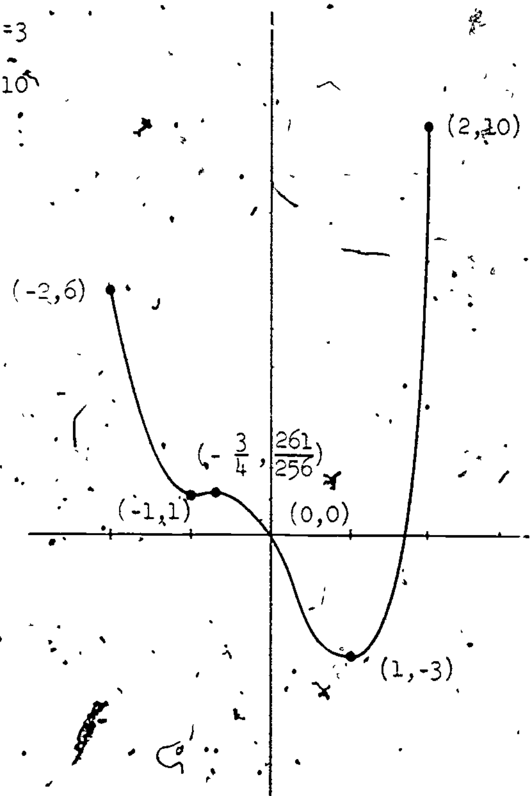
9. (a) $(-1,1)$, $(-\frac{3}{4}, \frac{261}{256})$, and $(1,-3)$

(b) $(-1,1)$ and $(1,-3)$

(c) $f(1) = -3$

(d) $f(2) = 10$

(e)



10. $(1,0)$ is a relative maximum

$(2,-2)$ is a point of inflection

$(3,-4)$ is a relative minimum

11. A(g), B(h), C(e), D(f)

12. (a) $(2,0)$ is a minimum point.

(b) $(2,0)$ is a point of inflection with horizontal tangent.

(c) $(2,0)$ is a minimum point.

13. (a) $(-2,13)$ is a relative maximum point.

$(1,-14)$ is a relative minimum point.

$(-\frac{1}{2}, -\frac{1}{2})$ is a point of inflection.

(b) $(-2,32)$ is a relative maximum point.

$(2,0)$ is a relative minimum point.

$(0,16)$ is a point of inflection.

(c) $(2,27)$ is a relative maximum point.

$(-1,0)$ is a relative minimum point.

$(\frac{1}{2}, \frac{27}{2})$ is a point of inflection.

(d) $(-1,4)$ is a relative maximum point.

$(1,0)$ is a relative minimum point.

$(0,2)$ is a point of inflection.

14. (a) $f' : x \to 3x^2 - 18x + 24$

(b) Relative maximum point is $(2,2)$.
Relative minimum point is $(4,-2)$.

(c) $f'' : x \to 6x - 18$

(d) $(3,0)$

(e) $f'(3) = -3$

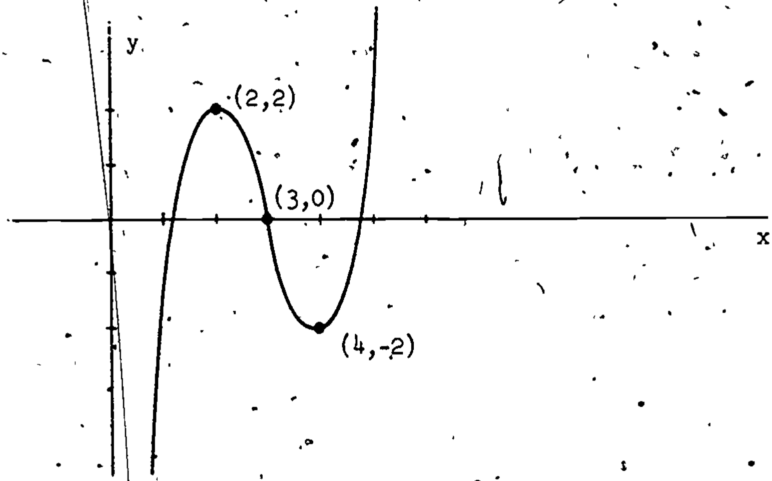
(f) $f'(x) = 3x^2 - 18x + 24 = 3(x - 2)(x - 4)$

$$f'(3 + k) = 3(3 + k - 2)(3 + k - 4)$$
$$= 3(k + 1)(k - 1)$$
$$= 3(k^2 - 1)$$

$$f'(3 - k) = 3(3 - k - 2)(3 - k - 4)$$
$$= 3(-k + 1)(-k - 1)$$
$$= 3(k^2 - 1)$$

$$f'(3 + k) = f'(3 - k) = 3(k^2 - 1)$$

(g)



(h) The graph of f is symmetric with respect to the point of inflection $(3,0)$, since for every point $(3 + k, f(3 + k))$ on the graph of f there is a corresponding point $(3 - k, -f(3 + k))$; i.e.,

$$f(3 - k) = -f(3 + k)$$

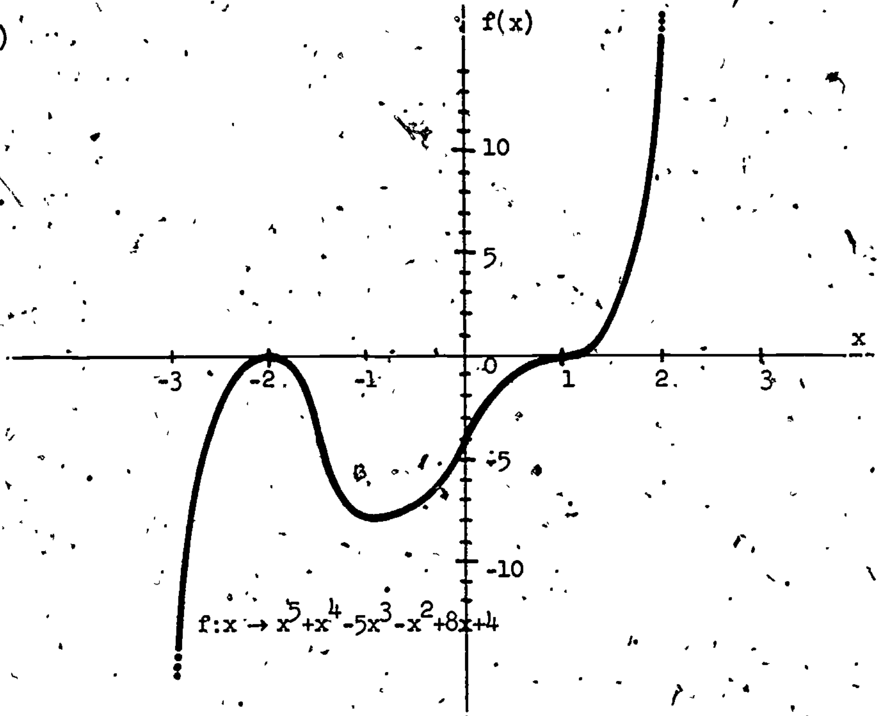
15. $f : x \to ax^3 + bx^2 + cx + d, \quad a \neq 0$

$f' : x \to 3ax^2 + 2bx + c$

$f'' : x \to 6ax + 2b$

The linear function f'' has a real zero $-\frac{b}{3a}$ since $a \neq 0$. Furthermore the sign of $f''(x)$ for $x < -\frac{b}{3a}$ is different from $f''(x)$ for $x > -\frac{b}{3a}$.

16. (a)



(b) The zeros of f are 1 (of multiplicity three) and -2 (of multiplicity two).

(c) At $(-2,0)$ the graph is flexed downward, at $(1,0)$ the graph is flexed neither upward nor downward.

(d) Same as part (c).

17. $(-1, -\frac{8}{15})$, $(1, \frac{8}{15})$

18. $(\frac{5}{2}, -\frac{15}{16})$ is a relative minimum point.

19. $(0,0)$ and $(-1,-11)$ are points of inflection.

20. (a) $f' = g$

(b) (i) This is an x-intercept point for each of the functions f and g. A zero of g is -1 since $g(-1) = 0$. Because $(-1,0)$ is also a relative maximum point on the graph of f, we know that -1 is a zero of f of multiplicity two.

(ii) This point $(0,-2)$ is a point of inflection of the graph of f, as well as the y-intercept point of that graph.

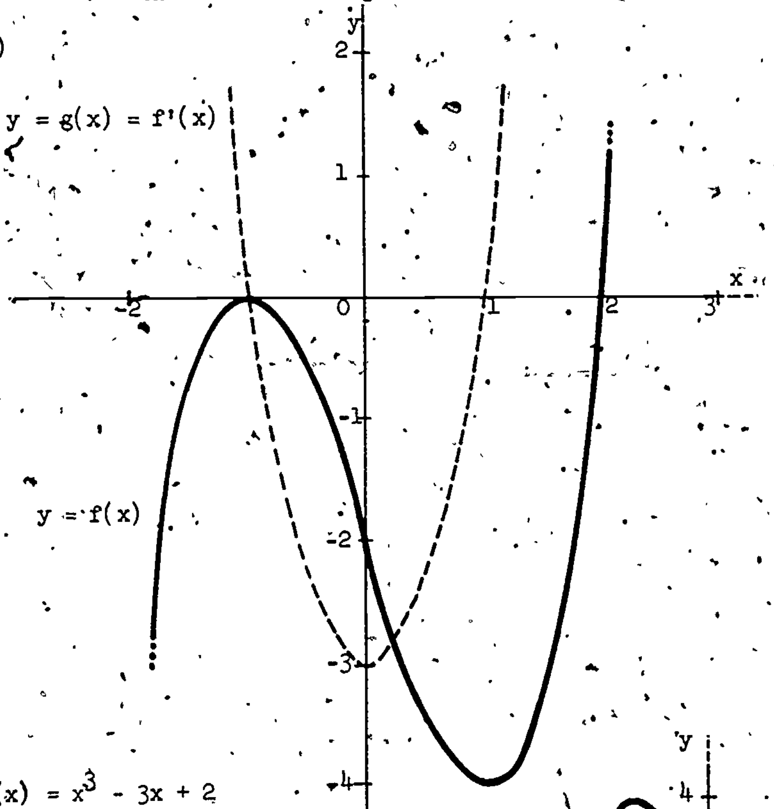
(iii) This y-intercept point $(0,3)$ on the graph of g is the minimum point of the parabola.

(iv) This x-intercept point (1,0) of the graph of g indicates that the other zero of g is 1.

(v) This point (1,-4) is a relative minimum point on the graph of f.

(vi) This x-intercept point (2,0) of the graph of f indicates that the remaining zero of f is 2.

(c)



21. $f(x) = x^3 - 3x + 2$

$= (x - 1)^2(x + 2)$

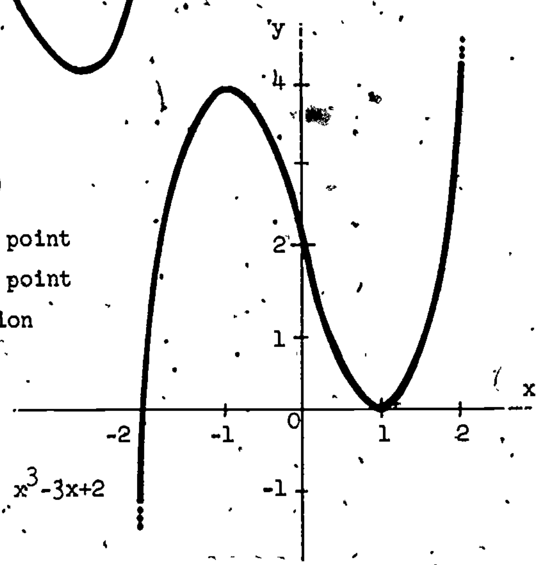
(a) 2, 1 (of multiplicity two)

(b) (1,0) is relative minimum point

(-1,4) is relative maximum point

(0,2) is point of inflection

(c) graph:



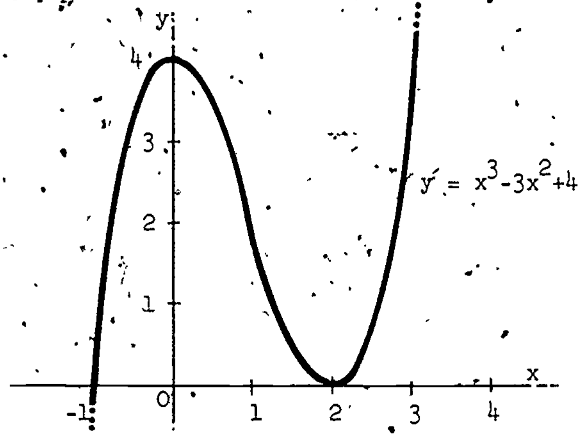
22. (a) Zeros are 2 (of multiplicity two) and -1.

(b) $(0,4)$ is a relative maximum point

$(2,0)$ is a relative minimum point

$(1,2)$ is the point of inflection

(c) graph:



23. We have $f'(x) = 2Ax + B$ and

$$f''(x) = 2A.$$

If the point $(x_0,f(x_0))$ is a point of inflection we must satisfy the condition

$$f''(x_0) = 2A = 0.$$

But we have the restriction $A \neq 0$. Therefore, the graph of $f : x \to Ax^2 + Bx + C$, $A \neq 0$ has no point of inflection.

24. We have $f'(x) = 3x^2 + 6x - 4$ and

$$f''(x) = 6x + 6.$$

To find the point of inflection we let $f''(x) = 6x + 6 = 0$ and get $x = -1$. By the usual arguments we decide that the point of inflection is $(-1,3)$. The slope of the tangent at $(-1,3)$ is $f'(-1) = -7$. An equation of the tangent is

$$y = -7x - 4.$$

Solutions Exercises 2-10

1. (a) $f'(0.8) = 4.52$

   (b) $y = -0.048 + 4.52(x - 0.8)$.

   (c) 0.81

2. (a) $f'(1.5) = 6.75$

   (b) $y = 0.375 + 6.75(x - 1.5)$

   (c) $x \approx 1.444$

3. (a) $\sqrt{2}$

   (b) $f(1) < 0$ and $f(2) > 0$

   (c) Use the method of Example 2-11b to obtain $\sqrt{2} \approx 1.414$.

4. (a) $f(0) > 0$ and $f(1) < 1$

   (b) $f' : x \to 3x^2 - 12$

   (c) $f(0) = 1$ and $f'(0) = -12$

   (d) $x_2 = 0 - \frac{1}{-12} \approx 0.08$

   (e) With $x_2 \approx 0.08$ we use synthetic substitution:

$$\begin{array}{rrrr|l} 1 & 0 & -12 & 1 & 0.08 \\ & .08 & .0064 & -.959488 & \\ \hline 1 & .08 & -11.9936 & .040512 \approx f(.08) & \end{array}$$

$$\begin{array}{rrr|l} 3 & 0 & -12 & 0.08 \\ & .24 & .0192 & \\ \hline 3 & .24 & -11.9808 \approx f'(.08) & \end{array}$$

   (f) $x_3 = 0.08 - \frac{0.040512}{-11.9808} \approx 0.08 + 0.0034 \approx 0.083$

5. $f(x) = x^3 - 3x^2 + 2$ $f'(x) = 3x^2 - 6x$

$f(2) = -2$

$f(3) = 2$ try $x_1 = 2.7$

$$\begin{array}{rrrr|l} 1 & -3 & 0 & 2 & \underline{2.7} \\ & 2.7 & -.81 & -2.187 & \\ \hline 1 & -0.3 & -.81 & -0.187 & = f(2.7) \end{array}$$

$$\begin{array}{rrr|l} 3 & -6 & 0 & \underline{2.7} \\ & 8.1 & 5.67 & \\ \hline 3 & 2.1 & 5.67 & = f'(2.7) \end{array}$$

$$x_2 = 2.7 - \frac{-0.187}{5.67} \approx 2.7 + .033 \approx 2.73$$

6. Consider the function $f : x \to x^3 + 3x - 7$

$f(x) = x^3 + 3x - 7$ $f'(x) = 3x^2 + 3$

$f(1) = -3$

$f(2) = 7$ take $x_1 = 1.4$

$$\begin{array}{rrrr|l} 1 & 0 & 3 & -7 & \underline{1.4} \quad f'(1.4) = 3(2.96) = 8.88 \\ & 1.4 & 1.96 & 6.944 & \\ \hline 1 & 1.4 & 4.96 & -0.056 & = f(1.4) \end{array}$$

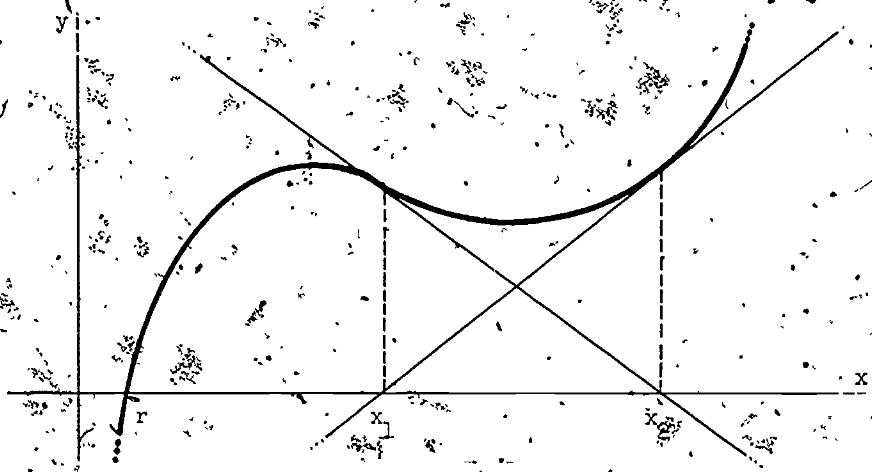
$$x_2 = 1.4 - \frac{-0.056}{8.88} \approx 1.406(3)$$

$$\begin{array}{rrrr|l} 1 & 0 & 3 & -7 & \underline{1.41} \quad f'(1.41) = 8.9643 \\ & 1.41 & 1.9881 & 7.033221 & \\ \hline 1 & 1.41 & 4.9881 & .033221 & = f(1.41) \end{array}$$

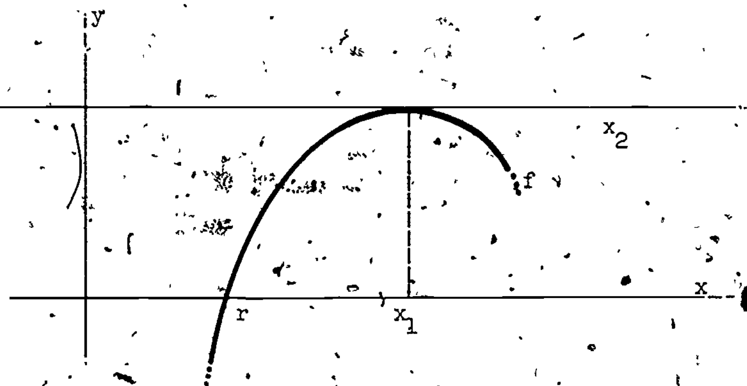
$$x_3 = 1.41 - \frac{0.033221}{8.9643} \approx 1.406$$

To two decimal places a zero of the function f is 1.41. Therefore, an approximate solution of $x^3 + 3x = 7$ correct to two decimal places is 1.41.

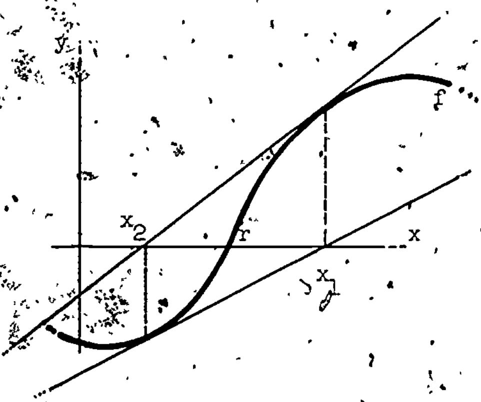
7. (a)



or



(b)



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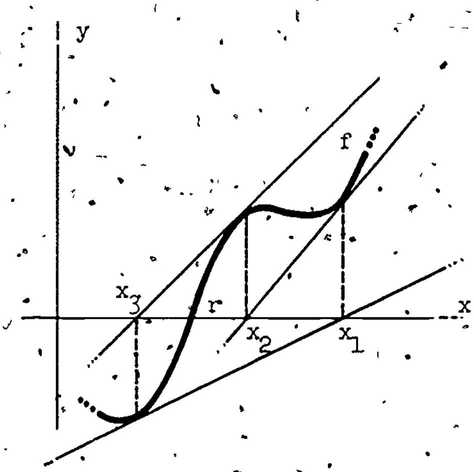


(c) It is sufficient to require that $f'(x)$ and $f''(x)$ do not change their respective signs over the interval containing the root r and its first and second approximations, x_1 and x_2.

1. n

2. $K = \frac{1}{1 \times 2 \times 3 \times 4 \times 5} = \frac{1}{5!} = \frac{1}{120}$

3. (a) $g(x) = x^2 + 2x + 1$, $f(2) = 2$

   (b) $p(x) = x + 4$, $g(2) = 9$

   (c) $q(x) = 1$, $p(2) = 6$

   (d) $q(2) = 1$

   (e) We have

$$f(x) = (x - 2)g(x) + f(2),$$
$$g(x) = (x - 2)p(x) + g(2),$$
$$p(x) = (x - 2)q(x) + p(2),$$
$$q(x) = q(2).$$

   Therefore,

$$\begin{aligned} f(x) &= (x - 2)[(x - 2)p(x) + g(2)] + f(2) \\ &= (x - 2)\{(x - 2)[(x - 2)q(x) + p(2)] + g(2)\} + f(2) \\ &= (x - 2)\{(x - 2)[(x - 2)q(2) + p(2)] + g(2)\} + f(2). \end{aligned}$$

   (f) From part (e) we have

$$f(x) = q(2)(x - 2)^3 + p(2)(x - 2)^2 + g(2)(x - 2) + f(2)$$

   Substituting from parts (a) through (d) we get

$$f(x) = 1(x - 2)^3 + 6(x - 2)^2 + 9(x - 2) + 2.$$

   Therefore, $A = 1$, $B = 6$, $C = 9$, and $D = 2$.

   (g) $f' : x \to 3x^2 - 3$
   $f'' : x \to 6x$
   $f''' : x \to 6$

   (h) $f(2) = 2$, $f'(2) = 9$, $f''(2) = 12$, $f'''(2) = 6$

   (i) $\frac{f(2)}{0!} = 2$, $\frac{f'(2)}{1!} = 9$, $\frac{f''(2)}{2!} = 6$, $\frac{f'''(2)}{3!} = 1$

(j) $A = \frac{f'''(2)}{3!}$, $B = \frac{f''(2)}{2!}$, $C = \frac{f'(2)}{1!}$, $D = \frac{f(2)}{0!}$

4\. (a) $G(2.1) = (2.1)^3 - 3(2.1) = 9.261 - 6.3 = 2.961$

$f(2.1) = 2$

$g(2.1) = 2 + 0.9 = 2.9$

$h(2.1) = 2 + 0.9 + 0.06 = 2.96$

$F(2.1) = 2 + 0.9 + 0.06 + 0.001 = 2.961$

(b) $h : x \to 2 + 9(x - 2) + 6(x - 2)^2$

(c) $g : x \to 2 + 9(x - 2)$

(d) $x \to -3(x + 1)^2 + 2$

(e) $x \to a^3 - 3a + (3a^2 - 3)(x - a) + 3a(x - a)^2$

5\. (a) $F' : x \to 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

$F'' : x \to 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$F''' : x \to 1 + x + \frac{x^2}{2!}$

$F^{(4)} : x \to 1 + x$

(b) $f' : x \to 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \frac{x^{10}}{10!}$

$f'' : x \to -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!}$

$f''' : x \to -1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!}$

$f^{(4)} : x \to x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

(c) $g' : x \to -x + \frac{x^3}{3!} - \frac{x^5}{5!} + \frac{x^7}{7!} - \frac{x^9}{9!}$

$g'' : x \to -1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{x^6}{6!} - \frac{x^8}{8!}$

$g''' : x \to x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!}$

$g^{(4)} : x \to 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$

6. (a)

7	0	0	-2	3	-1
	-7	7	-7	9	
7	-7	7	-9	12	

7	-7	7	-9	-1
	-7	14	-21	
7	-14	21	-30	

7	-14	21	-1
	-7	21	
7	-21	42	

7	-21	-1
	-7	
7	-28	

(b) $f: x \to 12 - 30(x+1) + 42(x+1)^2 - 28(x+1)^3 + 7(x+1)^4$

This agrees with the result of Example 2-11a.

(c) linear $x \to 12 - 30(x+1)$

quadratic $x \to 12 - 30(x+1) + 42(x+1)^2$

cubic $x \to 12 - 30(x+1) + 42(x+1)^2 - 28(x+1)^3$

(d) falling and flexed upward

7. $f: x \to ax^2 + bx + c$

$f': x \to 2ax + b$

$f'': x \to 2a$

$f''': x \to 0$

Teacher's Commentary

Chapter 3

CIRCULAR FUNCTIONS

In Chapter 1 we considered the algebra of polynomial functions; then in Chapter 2, the calculus of the same functions. We continue this "function - calculus of function" sequence in Chapters 3 and 4 respectively.

Unlike the polynomial functions considered in the first two chapters we now discuss functions which have the property that their function values repeat themselves in the same order at regular intervals over the domain -- periodic functions. We shall consider several types of periodic functions in the course of the text, but in Chapters 3 and 4 we invite the student to focus his attention on the circular functions. You may wish to refer the student to Appendix 1 for additional work with circular functions as well as discussion of other periodic functions. We do not intend for this chapter to be (or even to review) a course in the solution of triangles. We expect that this aspect of trigonometry will have been studied in a previous course. That is not unconditionally necessary, however. We do hope to cultivate an attitude that will prove useful to the student. We want the student eventually to be able to think of the circular functions as purely numerical functions apart from the ideas of the geometry of the triangle.

If a student is able to think about the circular functions analytically we believe that he can then develop systematic computational techniques for their use and greatly expand their range of application. For example, the property of the circular functions essential for higher analysis is their periodicity, a property to which the considerations of elementary geometry and trigonometry scarcely point. The role of periodicity in our understanding of natural phenomena is profound. The circular functions correspond directly to the simplest periodic motions, the turning of a wheel or the motion of a particular transcribing a circle at uniform speed. Yet combinations of these same elementary circular functions can be used to represent the most intricate periodic phenomena.

In Section 3-4 it would seem useful to point out that the simplest periodic motion is that of a wheel rotating on its axle. Each complete turn of the wheel brings it back to the position it held at the beginning. After a point of the wheel traverses a certain distance in its path about the axle, it returns to its initial position and retraces its course again. The distance traversed by the point in a complete cycle of its motion is again a period, a period measured in units of length instead of units of time. If it should happen that equal lengths are traversed in equal times, the motion becomes periodic in time as well and the wheel can be used as a clock.

The concept of circular function is based upon ideas (like the idea of limit) which are not usually stated precisely until a more rigorous course than this. Nonetheless, the circular functions are too important to neglect. Beginning with this chapter we use them freely, assuming all the properties which are familiar to the student from his earlier courses. Without concern for a strictly logical derivation of these properties, we shall argue geometrically and intuitively to obtain those properties central to this course.

We use the uv-plane when we define sine and cosine in terms of the unit circle, but later (Section 3-3) we display the graphs of $x \to \sin x$ and $x \to \cos x$ on the xy-plane. We do this to avoid the confusion which might result if we were to teach the student to try to visualize x as the measure of the horizontal axis on the plane of the unit circle and at the same time as the measure of length of the circular arc. We believe that the use of u and v is pedagogically more satisfactory than trying to get x to wear two hats in Section 3-1 and in the transition from Section 3-1 to Section 3-3.

A more exact way of defining sine and cosine is by a composition of two functions, one from the set of real numbers to the set of geometric points on the unit circle and the other from the set of points on the circle to the set of real numbers. Thus, if x is a real number and if P is a point on the unit circle, we have a function

$$f : x \to P$$

and another function

$$g : P \to \cos x,$$

from which we obtain the composite function

$$gf : x \to \cos x.$$

The sine function can be similarly defined. We believe, however, that the way in which we have handled it in the text, while possibly less rigorous, is certainly easier to teach and is perfectly adequate for our purposes.

Our experience suggests that the fact that cosine and sine are functions from real numbers to real numbers needs to be emphasized. Although we have used the concept of arc length, sine and cosine can be completely divorced from any geometric considerations. They are functions on the set of real numbers in the same sense as polynomial or exponential functions. Traditionally when one spoke of sin A, a student thought of A as the measure of an angle. He may have thought of A as being the degree measure or radian measure of an angle, but the idea that A need have no connection with an angle has been foreign to many students in our experience.

There are three closely related, though distinct, kinds of trigonometric functions passing under the same name. First, there are the trigonometric functions of geometrical objects, namely, the angles introduced in geometry. Then, when we introduce an angle measure, the functions are functions of a real variable. The real functions depend upon the measure of the angles. Thus, the numerical functions obtained by measuring angles in degrees (a relic of the Babylonian sexagesimal numeration) are not the same as the functions obtained by measuring the angle in radians. Since an angle measured by x degrees is measured by $\frac{\pi}{180}x$ radians, a trigonometric function, say the sine function, defined in terms of degree measure is related to the corresponding trigonometric function defined in terms of radian measure by

$$\sin^{\circ} x = \sin \frac{\pi}{180} x .$$

Central angles are related to the unit circle as the chapter progresses. Having begun with functions connected with an arc length x units, we expand our consideration to deal with sine and cosine as time functions, using ωt, where ω (as is the case in most scientific applications) is the measure of angular velocity.

When we use a graph to enhance the student's understanding of a function which maps real numbers into real numbers, we give a true picture of the function only when we use the same scale on both axes. However, it is sometimes desirable to distort the graph by using different scales in order to show important details which might otherwise be indistinct or confused, and when we graph an equation which describes the relationship between two physical quantities, the question of equal scales may be meaningless. If the pressure p at time t is given by an equation of the form $p = A \cos (\omega t + \beta)$, we cannot use the same scale on the p-axis as on the t-axis because there is no common measure for time and pressure. Because this situation is one of common occurrence in applications of the circular functions, we have not always insisted on the equal-scales principle.

Solutions Exercises 3-1a

1. (a) 120°
   (b) 30°
   (c) -120°
   (d) 210°
   (e) 360°
   (f) 150°
   (g) 480°
   (h) 648°
   (i) 585°
   (j) $(\frac{270}{\pi})^\circ \approx 86^\circ$
   (k) $(\frac{126}{\pi})^\circ \approx 40^\circ$
   (ℓ) $(\frac{468}{\pi})^\circ \approx 149^\circ$

2. (a) $\frac{3\pi}{2}$
   (b) $-\frac{\pi}{6}$
   (c) $\frac{3\pi}{4}$
   (d) $\frac{8\pi}{3}$
   (e) $\frac{13\pi}{12}$
   (f) $-\frac{7\pi}{12}$
   (g) $\frac{9\pi}{2}$
   (h) $\frac{19\pi}{18}$
   (i) $\frac{\pi}{10}$
   (j) $\frac{\pi}{450} \approx .007$ rad.
   (k) $9\pi \approx 28.3$ rad.
   (ℓ) $(\frac{18}{\pi})^\circ = (\frac{18}{\pi})(\frac{\pi}{180}) = .1$ rad.

3. (a) π, 2π
   (b) $(n-2)\pi$, 2π
   (c) $\frac{11\pi}{12}$
   (d) $\frac{\pi}{2}$, $\frac{5\pi}{6}$
   (e) 3π, $\frac{23\pi}{3}$

4. (a) $(\frac{1}{2}, -\frac{\sqrt{3}}{2})$
   (b) $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$
   (c) $(-\frac{1}{2}, -\frac{\sqrt{3}}{2})$
   (d) $(-1,0)$

5. (a) $\frac{\pi}{6}$
   (b) $\frac{2\pi}{3}$
   (c) $\frac{2\pi}{7}$
   (d) $\frac{13\pi}{12}$

6. (a) (i) $\frac{17\pi}{4} = 2(2\pi) + \frac{\pi}{4} = 3(2\pi) + (-\frac{7\pi}{4})$

(ii) $\frac{17\pi}{4} = 4(\pi) + \frac{\pi}{4} = 5(\pi) + (-\frac{3\pi}{4})$

(iii) $\frac{17\pi}{4} = 8(\frac{\pi}{2}) + \frac{\pi}{4} = 9(\frac{\pi}{2}) + (-\frac{\pi}{4})$

(b) (i) $-\frac{31}{12}\pi = -1(2\pi) + (-\frac{7}{12}\pi) = -2(2\pi) + (\frac{17}{12}\pi)$

(ii) $-\frac{31}{12}\pi = -2(\pi) + (-\frac{7\pi}{12}) = -3(\pi) + (\frac{5\pi}{12})$

(iii) $-\frac{31}{12}\pi = -5(\frac{\pi}{2}) + (-\frac{\pi}{12}) = -6(\frac{\pi}{2}) + (\frac{5\pi}{12})$

(c) (i) $\frac{28}{5}\pi = 2(2\pi) + (\frac{8\pi}{5}) = 3(2\pi) + (-\frac{2\pi}{5})$

(ii) $\frac{28}{5}\pi = 5(\pi) + (\frac{3\pi}{5}) = 6(\pi) + (-\frac{2\pi}{5})$

(iii) $\frac{28}{5}\pi = 11(\frac{\pi}{2}) + (\frac{\pi}{10}) = 12(\frac{\pi}{2}) + (-\frac{2\pi}{5})$

(d) (i) $-\frac{23\pi}{8} = -2(2\pi) + (\frac{9}{8}\pi) = -1(2\pi) + (-\frac{7\pi}{8})$

(ii) $-\frac{23}{8}\pi = -3(\pi) + (\frac{\pi}{8}) = -2(\pi) + (\frac{-7\pi}{8})$

(iii) $-\frac{23\pi}{8} = -6(\frac{\pi}{2}) + (\frac{\pi}{8}) = -5(\frac{\pi}{2}) + (\frac{-3\pi}{8})$

7. (a)

$Q_1\ (\frac{\sqrt{3}}{2}, \frac{1}{2})$	$Q_5\ (-\frac{\sqrt{3}}{2}, \frac{1}{2})$	$Q_9\ (0,-1)$
$Q_2\ (\frac{1}{2}, \frac{\sqrt{3}}{2})$	$Q_6\ (-1,0)$	$Q_{10}(\frac{1}{2}, -\frac{\sqrt{3}}{2})$
$Q_3\ (0,1)$	$Q_7\ (-\frac{\sqrt{3}}{2}, -\frac{1}{2})$	$Q_{11}(\frac{\sqrt{3}}{2}, -\frac{1}{2})$
$Q_4\ (-\frac{1}{2}, \frac{\sqrt{3}}{2})$	$Q_8\ (-\frac{1}{2}, -\frac{\sqrt{3}}{2})$	$Q_{12}(1,0)$

(b) Every other one; i.e., $Q_2, Q_4, Q_6, Q_8, Q_{10}, Q_{12}$ have the same coordinates respectively to $P_1, P_2, \ldots, P_6$.

(c) Q_3, Q_6, Q_9, Q_{12}

(d) You cannot read off values of functions in such a way. Countless counter examples can be readily seen; e.g.

$$\frac{1}{2} = \sin\frac{\pi}{6} \neq \frac{\sin\frac{\pi}{3} + \sin 0}{2} = \frac{\sqrt{3}}{4}.$$

8. (a) $K_1\ (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ $\qquad$ $K_5\ (-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

$K_2\ (0,1)$ $\qquad$ $K_6\ (0,-1)$

$K_3\ (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ $\qquad$ $K_7\ (\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}})$

$K_4\ (-1,0)$ $\qquad$ $K_8\ (1,0)$

(b) (1) $\frac{1}{\sqrt{2}}$ $\qquad$ (6) $-\frac{1}{\sqrt{2}}$

(2) $\frac{1}{\sqrt{2}}$ $\qquad$ (7) $-\frac{1}{\sqrt{2}}$

(3) $-\frac{1}{\sqrt{2}}$ $\qquad$ (8) $-\frac{1}{\sqrt{2}}$

(4) -1 $\qquad$ (9) $-\frac{1}{\sqrt{2}}$

(5) $\frac{1}{\sqrt{2}}$ $\qquad$ (10) $\frac{1}{\sqrt{2}}$

9. (a) $\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}$

$\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}$

Quadrant I: $\left[\begin{array}{l}\text{sine positive}\\ \text{cosine positive}\end{array}\right.$

$\sin\frac{\pi}{6} = \cos\frac{\pi}{3}$; $\sin\frac{\pi}{3} = \cos\frac{\pi}{6}$; $\sin\frac{\pi}{4} = \cos\frac{\pi}{4}$.

(b) $\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{\sqrt{3}}{2}$

$-\frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{2}$

Quadrant II: $\left[\begin{array}{l}\text{sine positive}\\ \text{cosine negative}\end{array}\right.$

(c) $-\frac{1}{2}, -\frac{1}{\sqrt{2}}, -\frac{\sqrt{3}}{2}$

$-\frac{\sqrt{3}}{2}, -\frac{1}{\sqrt{2}}, -\frac{1}{2}$

Quadrant III: $\left[\begin{array}{l}\text{sine negative}\\ \text{cosine negative}\end{array}\right.$

(d) $-\frac{1}{2}, -\frac{1}{\sqrt{2}}, -\frac{\sqrt{3}}{2}$

Quadrant IV: $\left[\begin{array}{l}\text{sine negative} \\ \text{cosine positive}\end{array}\right.$

$\frac{\sqrt{3}}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}$

(e) (i) Quadrants I and II. Quadrants III and IV.

(ii) Quadrants I and IV. Quadrants II and III.

(iii) II; IV; I; III.

10. (a) $P\left(\cos\frac{\pi}{6}, \sin\frac{\pi}{6}\right)$ or $P\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$

$P\left(2\cos\frac{\pi}{6}, 2\sin\frac{\pi}{6}\right)$ or $P(\sqrt{3}, 1)$

$P\left(4\sqrt{3}\cos\frac{\pi}{6}, 4\sqrt{3}\sin\frac{\pi}{6}\right)$ or $P(6, 2\sqrt{3})$

$P\left(R\cos\frac{\pi}{6}, R\sin\frac{\pi}{6}\right)$ or $P\left(\frac{R}{2}\sqrt{3}, \frac{R}{2}\right)$

(b) $T(2\cos x, 2\sin x)$

$T(7\cos x, 7\sin x)$

$T\left(\frac{3}{2}\cos x, \frac{3}{2}\sin x\right)$

$T(r\cos x, r\sin x)$

11. (a) Use similar triangles to show that $S = Rx$.

(b) $S = Rx = 4\left(\frac{\pi}{2}\right) = 2\pi$ inches ≈ 6.3 inches.

(c) $S = Rx = 15\left(\frac{\pi}{3}\right) = 5\pi$ inches ≈ 15.7 inches.

$S = Rx = 15\left(72 \cdot \frac{\pi}{180}\right) = 6\pi$ inches ≈ 18.8 inches.

(d) $S = Rx$

(i) $\frac{\pi}{6} = R\left(\frac{\pi}{6}\right)$ $\therefore$ $R = 1$

(ii) $\frac{\pi}{3} = R\left(\frac{\pi}{6}\right)$ $\therefore$ $R = 2$

(iii) $10 = R\left(\frac{\pi}{6}\right)$ $\therefore$ $R = \frac{60}{\pi} \approx 19.1$ (inches)

(iv) $x = R\left(\frac{\pi}{6}\right)$ $\therefore$ $R = \frac{6x}{\pi}$

(v) $3x = R\left(\frac{\pi}{6}\right)$ $\therefore$ $R = 3\left(\frac{6x}{\pi}\right) = \frac{18x}{\pi}$

(e) $S = Rx$

$x = Rx \quad \therefore \quad R = 1$

$2x = Rx \quad \therefore \quad R = 2$

$10x = Rx \quad \therefore \quad R = 10$

(f) $S = Rx$ or $R = \frac{S}{x} = \frac{\pi}{\frac{\pi}{10}} = 10$ (constant of proportionality)

$\therefore \quad S = 10x$

If $x = \pi$, $S = 10\pi$

If $x = \frac{2\pi}{3}$, $S = \frac{20\pi}{3}$

12. (a) $\frac{R^2}{2} = \frac{1}{2}$; 2; 8

(b) $A = \frac{a^2 m}{2}$

(c) doubled; halved; tripled

(d) $A = \frac{1}{2} R^2 x$, where x is arc measure

$= \frac{1}{2} R^2 \frac{S}{R}$, since arclength = radius $\times$ arc measure

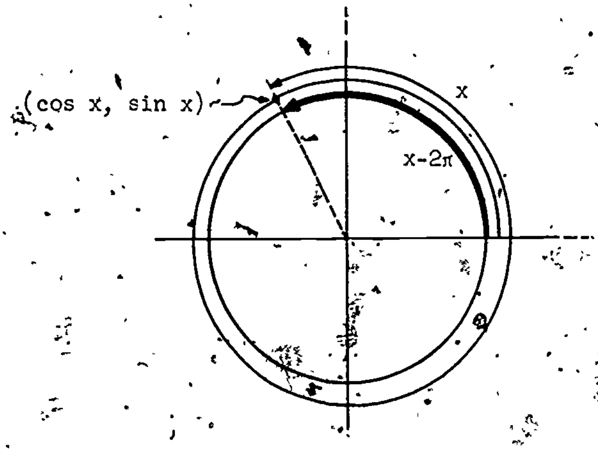
$= \frac{1}{2} R^2 S = (\frac{R}{2})s$

i.e., Area of sector is proportional to arclength where the constant of proportionality is $\frac{R}{2}$. $(S = R \cdot x$ or $x = \frac{S}{R})$

(e) $A = \frac{R}{2} S = \frac{18}{2}(12) = 108$ (in sq. in.)

(f) doubled; halved; tripled.

13. (a)



(b) For any integer n, $x + 2n\pi$ corresponds to n trips around the circumference of the unit circle (clockwise if $n < 0$ and counter-clockwise if $n \geq 0$). After the circumference has been traversed n times we arrive back at the same point, whose coordinates may be written either as

$$(\cos (x + 2n\pi), \sin (x + 2n\pi))$$

or

$$(\cos x, \sin x).$$

Hence,

$$\cos x = \cos (x + 2n\pi)$$

and

$$\sin x = \sin (x + 2n\pi).$$

Solutions Exercises 3-1b

Note: We use linear interpolation since close to a point a line serves to approximate the graph of sine or cosine.

1. (a) $\sin 0.73 \approx 0.6669$, $\cos 0.73 \approx 0.7452$

   (b) $\sin(-5.17) = \sin(-5.17 + 2\pi)$
   $\approx \sin 1.11 \approx 0.9857$
   $\cos(-5.17) \approx \cos 1.11 \approx 0.4447$

   (c) $\sin 1.55 \approx 0.9998$, $\cos 1.55 \approx 0.0208$

   (d) $\sin 6.97 = \sin(6.97 - 2\pi)$
   $\approx \sin 0.69 \approx 0.6365$
   $\cos 6.97 \approx \cos 0.69 \approx 0.7712$

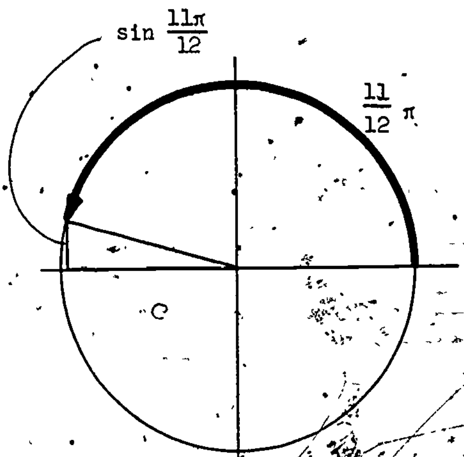
2. (a) $\sin x \approx 0.1098$, $x \approx 0.11$

   (b) $\cos x \approx 0.9131$, $x \approx 0.42$

   (c) $\sin x \approx 0.6518$, $x \approx 0.71$

   (d) $\cos x \approx 0.5403$, $x \approx 1.00$

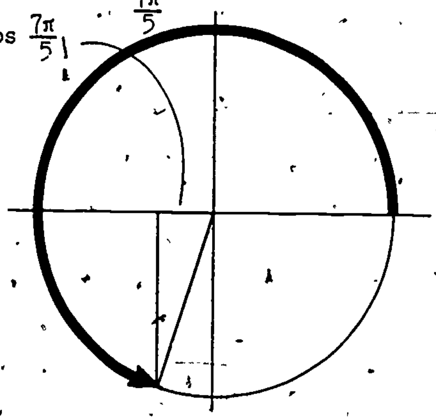
3. (a) $\sin \frac{11\pi}{12} = \sin \frac{\pi}{12}$
   $\approx \sin \frac{3.14}{12} \approx \sin 0.26$
   $\therefore \sin \frac{11\pi}{12} \approx .2571$



(b) $\cos \frac{7\pi}{5} = \cos(\pi + \frac{2\pi}{5})$

$= -\cos \frac{2\pi}{5}$

$\approx -\cos 1.256$



Interpolation

$\cos 1.25 = 0.3153$

$\cos 1.256 = \ ?$

$\cos 1.26 = 0.3058$

Difference = .0095

Either $\cos 1.256 \approx 0.3153 - .6(.0095) = .3153 - .0057 = .3096$

or $\cos 1.256 \approx 0.3058 + .4(.0095) = .3058 + .0038 = .3096$

$\therefore \cos \frac{7\pi}{5} \approx .3096$

(c) $\sin 11.50 = \sin[2(6.28 - 1.06)]$

$\approx \sin(-1.06)$

$\approx -\sin(1.06)$

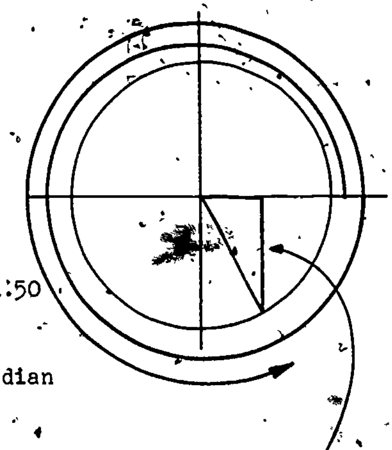
$\approx -.8724$

(d) $\cos 417^\circ = \cos(360^\circ + 57^\circ)$

$= \cos 57^\circ$

$\approx \cos 1$ Since $57^\circ \approx 1$ radian

≈ 0.5403



4. (a) $\sin .22 = .2182$, $\sin x = .2231$, $\sin .23 = .2280$; differences 49, 98

$\frac{49}{98} = .5$

$\therefore x = .225$

(b) $\cos .77 = 0.7179$, $\cos x = 0.7135$, $\cos .78 = 0.7109$; differences 14, 70

$\frac{14}{70} = \frac{2}{10}$

$\therefore x = .772$

(c) $\sin 1.05 = 0.8674$) 40) 50 $\quad \frac{40}{50} = .8$
$\sin x = 0.8714$
$\sin 1.06 = 0.8724$

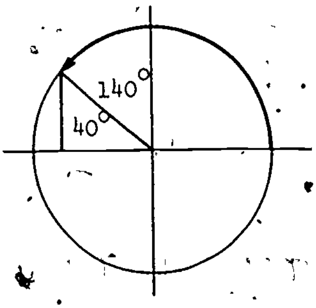
$\therefore x = 1.058$

(d) $\cos 1.39 = 0.1798$) 39) 98 $\quad \frac{39}{98} \approx .4$
$\cos x = 0.1759$
$\cos 1.40 = 0.1700$

$\therefore x = 1.394$

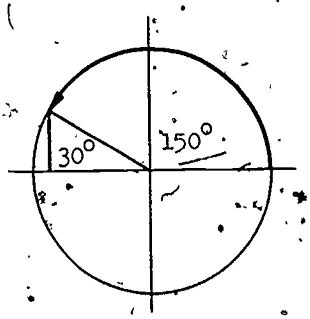
5. (a) $\sin 75^\circ \approx 0.966$

(b) Either $\cos 140^\circ = \cos(180^\circ - 40^\circ)$
$= \cos(-40^\circ) = \cos 40^\circ$
$x - .766$



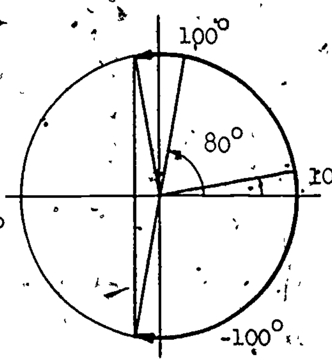
or $\cos 140^\circ = \cos(90^\circ + 50^\circ)$
$= -\sin 50^\circ$
$= -.766$

(c) $\sin 480^\circ = \sin(360^\circ + 120^\circ)$
$= \sin(120^\circ)$
$= \sin(180^\circ - 60^\circ)$
$= \sin 60^\circ$
$\approx .866$



(d) $\cos(-460^\circ) = \cos(-360^\circ - 100^\circ)$
$= \cos(-100^\circ)$
$= \cos(100^\circ)$

Either $\cos 100^\circ = \cos(180^\circ - 80^\circ) = -\cos 80^\circ$
or $\cos 100^\circ = \cos(90^\circ + 10^\circ) = -\sin 10^\circ$
i.e., $\cos 460^\circ \approx -0.174$



6. (a) If $\sin x = 0.574$, then $x = 35^\circ,\ 145^\circ$.

(b) If $\cos x = 0.643$, then $x = 50^\circ,\ 310^\circ$.

(c) If $\sin x = -0.819$, then $x = 235^\circ,\ 305^\circ$.

(d) If $\cos x = -0.087$, then $x = 175^\circ,\ 185^\circ$.

Solutions Exercises 3-2

1. (a) $f(3\pi) = f(\pi) = -1$

(b) $f(\frac{7\pi}{3}) = f(\frac{\pi}{3}) = \frac{1}{2}$

(c) $f(\frac{9\pi}{2}) = f(\frac{\pi}{2}) = 0$

(d) $f(\frac{25\pi}{6}) = f(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

(e) $f(-7\pi) = f(\pi) = -1$

(f) $f(-\frac{10\pi}{3}) = f(\frac{2\pi}{3}) = -\frac{1}{2}$

2. (a) $f(\pi) = 0$

(b) $f(\frac{\pi}{3}) = \frac{\sqrt{3}}{2}$

(c) $f(\frac{\pi}{2}) = 1$

(d) $f(\frac{\pi}{6}) = \frac{1}{2}$

(e) $f(\pi) = 0$

(f) $f(\frac{2\pi}{3}) = \frac{\sqrt{3}}{2}$

3. (a) $x = \frac{\pi}{4} + n\pi$

(b) $x = \frac{3\pi}{4} + n\pi$

(c) $x = n\pi$

(d) For all values of x.

4. (a) $$\frac{\sec\theta}{\sec\theta - \csc\theta} = \frac{\frac{1}{\cos\theta}}{\frac{1}{\cos\theta} - \frac{1}{\sin\theta}} = \frac{\frac{1}{\cos\theta}}{\frac{\sin\theta - \cos\theta}{\sin\theta\cdot\cos\theta}} = \frac{\sin\theta}{\sin\theta - \cos\theta}$$

(b) (i) $$\frac{\tan\theta + \sec\theta}{\sin\theta\cot\theta} = \frac{\frac{\sin\theta}{\cos\theta} + \frac{1}{\cos\theta}}{\sin\theta\,\frac{\cos\theta}{\sin\theta}} = \frac{\frac{1 + \sin\theta}{\cos\theta}}{\cos\theta} = \frac{1 + \sin\theta}{\cos^2\theta}$$

(ii) $$\frac{1 + \cot\theta}{\csc\theta} = \frac{1 + \frac{\cos\theta}{\sin\theta}}{\frac{1}{\sin\theta}} = \frac{\frac{\sin\theta + \cos\theta}{\sin\theta}}{\frac{1}{\sin\theta}} = \sin\theta + \cos\theta$$

$$\frac{1 + \tan\theta}{\sec\theta} = \frac{1 + \frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}} = \frac{\frac{\cos\theta + \sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}} = \sin\theta + \cos\theta$$

(iii) $\sin\theta \csc\theta = \sin\theta \dfrac{1}{\sin\theta} = 1$

$\cos\theta \sec\theta = \cos\theta \dfrac{1}{\cos\theta} = 1$

$\tan\theta \cot\theta = \tan\theta \dfrac{1}{\tan\theta} = 1$

5. (a) (i) $\cos^2\theta - \sin^2\theta = \cos^2\theta - (1 - \cos^2\theta) = 2\cos^2\theta - 1$

$\cos^2\theta - \sin^2\theta = (1 - \sin^2\theta) - \sin^2\theta = 1 - 2\sin^2\theta$

(ii) $\tan\theta + \cot\theta = \dfrac{\sin\theta}{\cos\theta} + \dfrac{\cos\theta}{\sin\theta} = \dfrac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta} = \dfrac{\csc\theta}{\cos\theta}$

(iii) $\dfrac{1}{1+\cos\theta} + \dfrac{1}{1+\cos\theta} = \dfrac{1+\cos\theta + 1 - \cos\theta}{1-\cos^2\theta} = \dfrac{2}{\sin^2\theta} = 2\csc^2\theta$

(iv) $\cot\theta \csc\theta = \dfrac{\cos\theta}{\sin\theta} \cdot \dfrac{1}{\sin\theta} = \dfrac{\cos\theta}{\sin^2\theta}$

$\dfrac{1}{\sec\theta - \cos\theta} = \dfrac{1}{\dfrac{1}{\cos\theta} - \cos\theta} = \dfrac{\cos\theta}{1 - \cos^2\theta} = \dfrac{\cos\theta}{\sin^2\theta}$

(b) (i) $(1 - \sin^2\theta)\sec^2\theta = \cos^2\theta \sec^2\theta = 1$

(ii) $(1 - \cos^2\theta)\csc^2\theta = \sin^2\theta \csc^2\theta = 1$

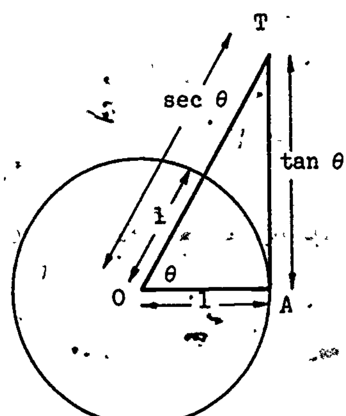
(iii) $\cot^2\theta(1 - \cos^2\theta) = \cot^2\theta \sin^2\theta = \cos^2\theta$

(iv) $\sec^2\theta(1 - \cos^2\theta) = \sec^2\theta \sin^2\theta = \dfrac{\sin^2\theta}{\cos^2\theta} = \tan^2\theta$

6. (a) $\sin^2\theta + \cos^2\theta = 1$

$\dfrac{\sin^2\theta}{\cos^2\theta} + 1 = \dfrac{1}{\cos^2\theta}$

$\tan^2\theta + 1 = \sec^2\theta$



(b) First, we need to find AT, the second coordinate of T:

$\tan\theta = \dfrac{AT}{OA} = AT$

Next, find the hypotenuse OT:

$\cos\theta = \dfrac{OA}{OT} = \dfrac{1}{OT}$; $\therefore$ $OT = \dfrac{1}{\cos\theta} = \sec\theta$

Now, we can read from the drawing; $1 + \tan^2\theta = \sec^2\theta$

(c) $\sin^2\theta + \cos^2\theta = 1$

$$1 + \frac{\cos^2\theta}{\sin^2\theta} = \frac{1}{\sin^2\theta}$$

$$1 + \cot^2\theta = \csc^2\theta$$

(d) (i) $\dfrac{\sec\theta}{\cos\theta} - \dfrac{\tan\theta}{\cot\theta} = \sec^2\theta - \tan^2\theta = 1$

(ii) $\sec^2\theta + \csc^2\theta = \csc^2\theta\left(\dfrac{\sec^2\theta}{\csc^2\theta} + 1\right) = \csc^2\theta(\tan^2\theta + 1)$

$$= \sec^2\theta\csc^2\theta$$

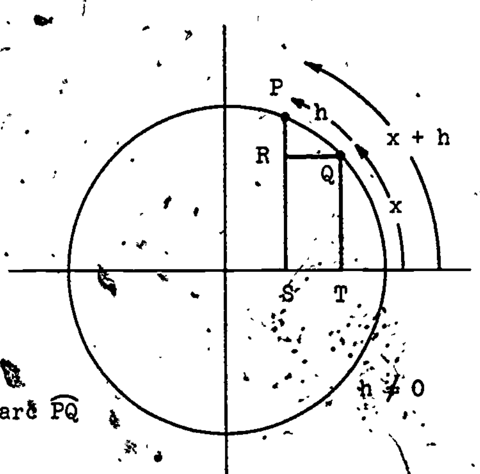
$$\sec^2\theta + \csc^2\theta = \frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} = \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta\cos^2\theta}$$

$$= \sec^2\theta\csc^2\theta$$

(iii) $\sin^2\theta(1 + \cot^2\theta) + \cos^2\theta(1 + \tan^2\theta)$

$$= \sin^2\theta\csc^2\theta + \cos^2\theta\sec^2\theta = 1 + 1 = 2$$

7. (a) $\sin(x + h) - \sin x = PS - QT$
$= PS - RS$
$= PR$
$< PQ$ (Since PQ is hypotenuse of rt. ΔPRQ.



(b) But chord $PQ < \text{arc } \widehat{PQ}$

$\therefore$ Using part (a)

$$\sin(x + h) - \sin x < PQ < \text{arc } \widehat{PQ}$$

or $\quad |\sin(x + h) - \sin x| < |h|$

(c) $\cos(x + h) - \cos x = OT - OS = ST = RQ < PQ < \widehat{PQ} = h$

$\therefore \quad |\cos(x + h) - \cos x| < |h|$

8. (a) $\Delta QMO \sim \Delta PNO$; $\therefore Q$ has coordinates $(-\cos x, -\sin x)$

(b) $\Delta RMO \sim \Delta PNO$; $\therefore R$ has coordinates $(\cos x, \sin x)$

9. (a) Since coordinates of P are cos x, sin x, coordinates of Q' are $(-\cos x, -\sin x)$.

(b) (i) $\cos(x - \pi) = \cos[-(\pi - x)]$

$= \cos(\pi - x)$ (Using (6))

$= -\cos x$ (See Number 8)

(ii) $\sin(x - \pi) = \sin[-(\pi - x)]$

$= -\sin(\pi - x)$ (Using (6))

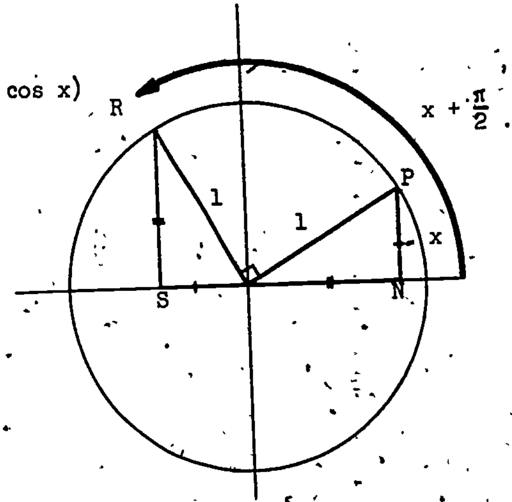
$= -\sin x$ (See Number 8)

10. $\Delta PNO \cong \Delta OSR$

$\therefore$ coordinates of R are $(-\sin x, \cos x)$

$\therefore \cos(x + \frac{\pi}{2}) = -\sin x$

$\sin(x + \frac{\pi}{2}) = \cos x$



11. (a) $\cos(x + \frac{\pi}{2}) = \sin[\frac{\pi}{2} - (x + \frac{\pi}{2})] = \sin(-x)$ (7)

$= -\sin x$ (6)

(b) (i) $\cos(x + \pi) = \sin[\frac{\pi}{2} - (x + \pi)] = \sin[-(\frac{\pi}{2} + x)]$ (7)

$= -\sin(\frac{\pi}{2} + x)$ (6)

$= -\cos[\frac{\pi}{2} - (\frac{\pi}{2} + x)] = -\cos(-x)$ (7)

$= -\cos x$ (6)

(ii) $\sin(x + \pi) = \cos[\frac{\pi}{2} - (x + \pi)] = \cos[-(\frac{\pi}{2} + x)]$ (7)

$= \cos(\frac{\pi}{2} + x)$ (6)

$= \sin[\frac{\pi}{2} - (\frac{\pi}{2} + x)] = \sin(-x)$ (7)

$= -\sin x$ (6)

(c) (i) $\cos(x - \pi) = \cos(\pi - x)$ (6)

$= \sin\left[\frac{\pi}{2} - (\pi - x)\right] = \sin\left[-\left(\frac{\pi}{2} - x\right)\right]$ (7)

$= -\sin\left(\frac{\pi}{2} - x\right)$ (6)

$= -\cos x$ (7)

(ii) $\sin(x - \pi) = -\sin(\pi - x)$ (6)

$= -\cos\left[\frac{\pi}{2} - (\pi - x)\right] = -\cos\left(x - \frac{\pi}{2}\right)$ (7)

$= -\cos\left(\frac{\pi}{2} - x\right)$ (6)

$= -\sin x$ (7)

12. (a)

(Rad.) x	$\cos x$	$1 - \cos x$	(Rad.) $\frac{x^2}{2}$
0	1	0	0
0.1	0.99995	0.00005	0.00005
0.15	0.98877	0.01123	0.01125
0.3	0.95534	0.04466	0.04500
0.5	0.87758	0.12242	0.12500
0.6	0.82534	0.17466	0.18000
0.7	0.76484	0.23516	0.24500
0.8	0.6967	0.3033	0.3200
0.9	0.6216	0.3784	0.4055
1.0	0.5403	0.4597	0.5000
1.2	0.3624	0.6376	0.7200
1.5	0.0707	0.9292	1.125
2.0	-0.4169	1.4169	2
4.0	-0.6524	1.6524	8
6.0	0.2771	.7229	18

Though the students have available only the 4-place tables, some of the values near $x = 0$ are given from 5-place Tables for the benefit of the teacher. When x is very small, this inequality is very nearly an equality. When $x = 0$, it is an equality.

$\cos 2 = \cos(1.57 + .43)$
$= -\sin(.43) = -.4169$

$\cos 4 = \cos(3.14 + .86)$
$= -\cos .86 = -.6524$

$\cos 6 = \cos[3(1.57) + 1.29]$
$= \sin 1.29 = .2771$

(b) Because $-1 \le \cos x \le +1$, $0 \le 1 - \cos x \le 2$, no matter how large x is. The value of $\frac{x^2}{2}$ continues to increase, e.g.

	$1 - \cos x$ Range ≈	$\frac{x^2}{2}$ Range ≈
In Quadrant I	0 to 1	0 to 1.23
Quadrant II	1 to 2	1.23 to 4.49
Quadrant III	2 to 1	4.49 to 11.10
Quadrant IV	1 to 0	11.10 to 19.74

Therefore this inequality is useful only when x is very near zero. When $x > 2$, it is obviously useless.

*13. (a) $\sin 2x = \sin(2x + 2\pi)$ from periodicity of sin,

$= \sin 2(x + \pi)$, and the period is π.

(b) $\sin \frac{1}{2}x = \sin (\frac{1}{2} x + 2\pi)$

$= \sin \frac{1}{2}(x + 4\pi)$, and the period is 4π.

(c) $\cos 4x = \cos(4x + 2\pi)$

$= \cos 4(x + \frac{\pi}{2})$, and the period is $\frac{\pi}{2}$.

(d) $\cos \frac{1}{2} x = \cos (\frac{1}{2} x + 2\pi)$

$= \cos \frac{1}{2}(x + 4\pi)$, and the period is 4π.

*14. If a is a period of cos, it must be true that

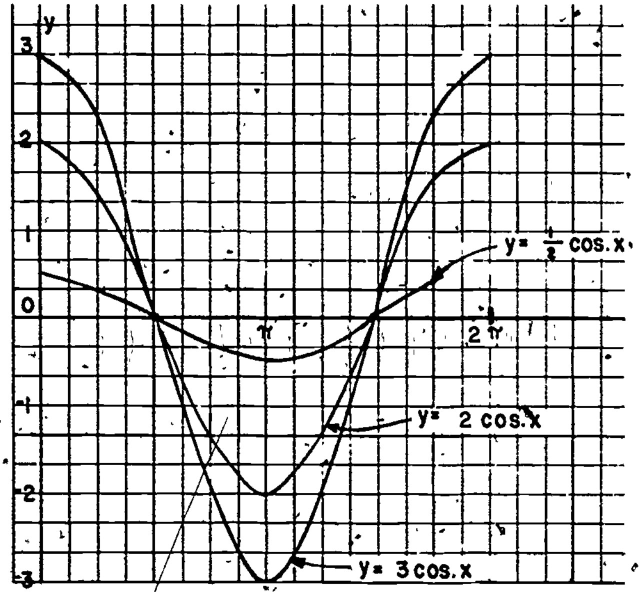
$$\cos(x + a) = \cos x$$

for all $x \in R$. In particular, it must be true if $x = 0$:

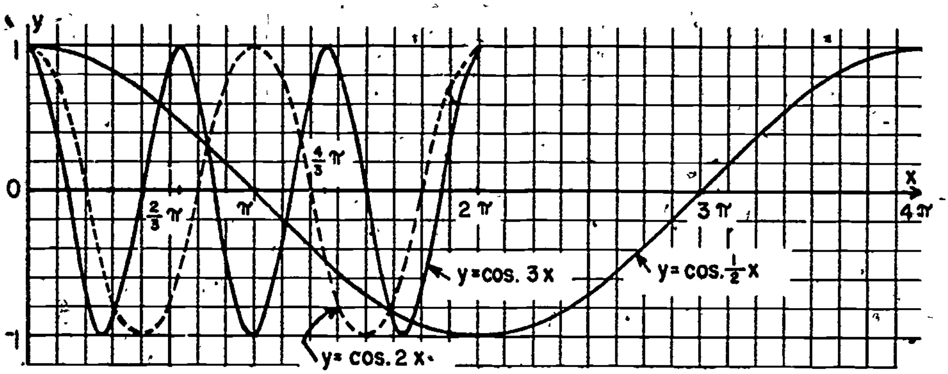
$$\cos a = \cos 0 = 1.$$

But the only point on the unit circle with abscissa 1 is $(1,0)$, which corresponds to $x = 0 + 2n\pi$. The proof for sin is similar; use $x = \frac{\pi}{2}$.

1.



2.

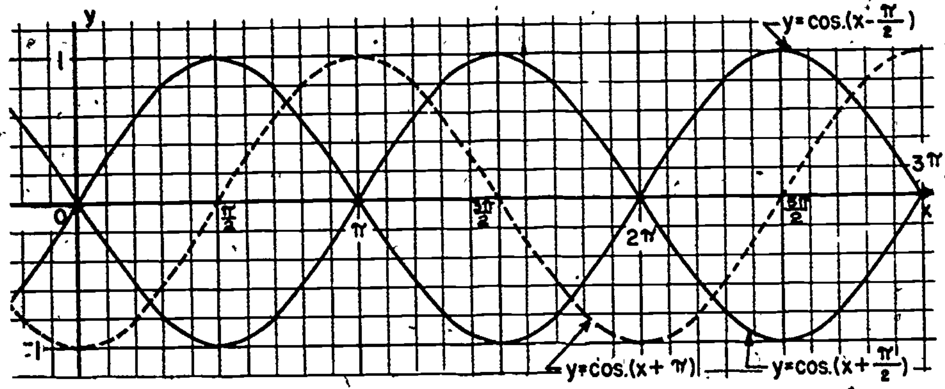


← 1 period →
$y = \cos 3x$

← 1 period →
$y = \cos 2x$

← 1 period →
$y = \cos \frac{1}{2} x$

3.

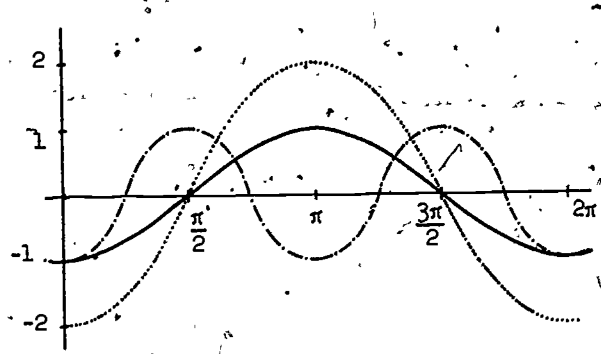


$\cos(x + \pi)$ or $\cos x$ shifted π units to the left.
period: 2π

$\cos(x + \frac{\pi}{2})$ or $\cos x$ shifted $\frac{\pi}{2}$ units left.
period: 2π

$\cos(x - \frac{\pi}{2})$ or $\cos x$ shifted $\frac{\pi}{2}$ units right.
period: 2π

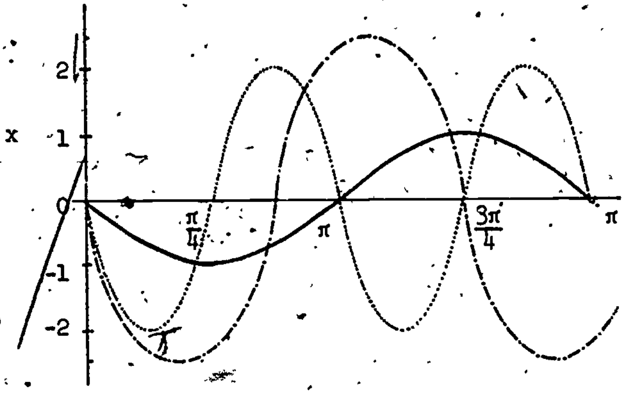
4. (a) ———— $y = -\cos x$
(b) .... $y = -2 \cos x$
(c) -.-.-. $y = -\cos 2x$



5. (a) ———— $y = -\sin 2x$

(b) .... $y = -2 \sin 4x$

(c) -.-.-.-. $y = -\frac{5}{2} \sin \frac{8}{3} x$

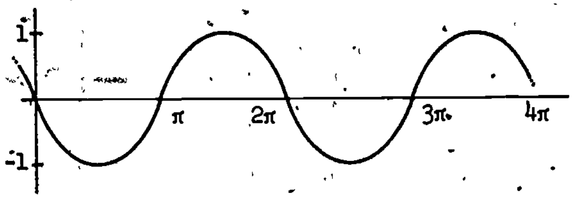


6. (a), (b), (c), (d), each is represented by the same graph, since

$$-\sin x = -\cos(x - \tfrac{\pi}{2})$$

$$= \sin(x + \pi)$$

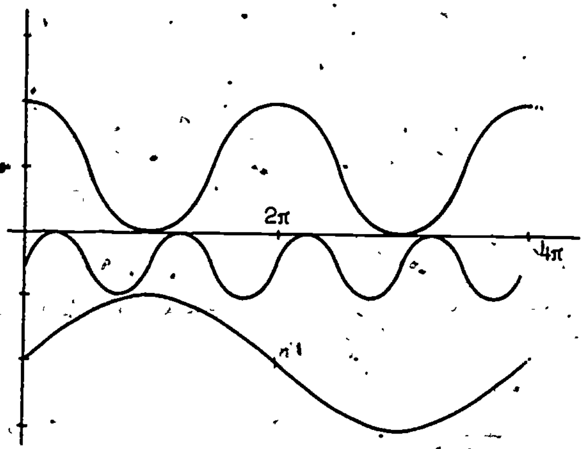
$$= \cos(x + \tfrac{\pi}{2})$$



7. (a) $y - 1 = \cos x$

(c) $y + \frac{1}{2} = \frac{1}{2} \sin 2x$

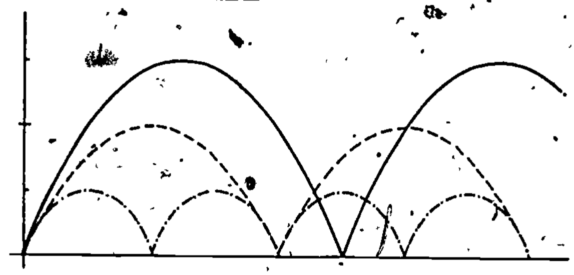
(b) $y + 2 = \sin \frac{x}{2}$



8. (a) ______ $y = |\sin x|$

(b) - - - - $y = \frac{1}{2}|\sin 2x|$

(c) .-.-.-. $y = \frac{3}{2}|\sin \frac{4}{5} x|$

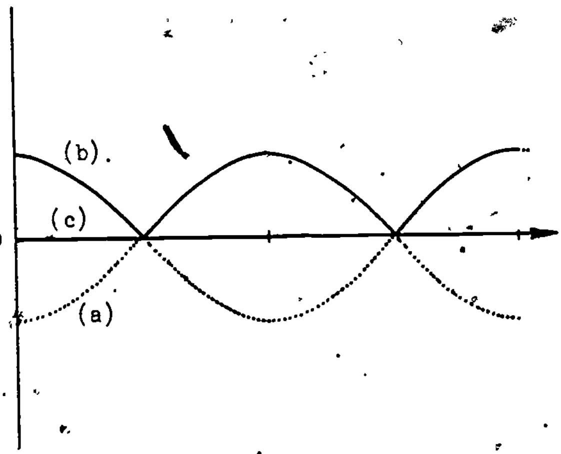


9. (b) ______ $y = |\sin(x - \frac{\pi}{2})|$

(c) ——→ Since $|\sin(x - \frac{\pi}{2})| = |\cos x|$,

$y = |\sin(x - \frac{\pi}{2})| - |\cos x| = 0$

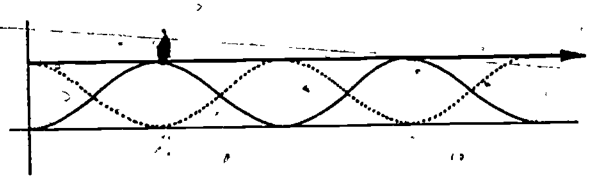
(a) ...... $y = -|\cos x|$



10. (a) ______ $y = \sin^2 x$

(b) .... $y = \cos^2 x$

(c) $y = 1$ Since $\sin^2 x + \cos^2 x = 1$

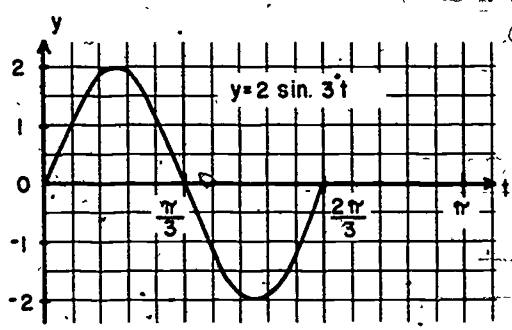


To the teacher: It is obviously simpler to graph $y = \sin^2 x$ and $y = \cos^2 x$ when the student knows $\sin^2 x = \frac{1}{2}(1 - \cos^2 x)$ and $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$. Then (a) and (b) can be written as $y - \frac{1}{2} = -\frac{1}{2} \cos 2x$ and $y - \frac{1}{2} = \frac{1}{2} \cos 2x$, respectively.

Solutions Exercises 3-4

We assume that the words domain and range are familiar to the student. Nevertheless, it may need to be noted that the range is twice the amplitude for these functions.

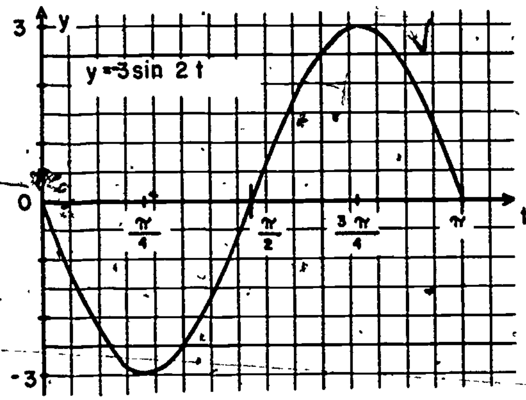
1. (a)



The period is $\frac{2\pi}{3}$.

The range is $-2 \leq y \leq 2$.

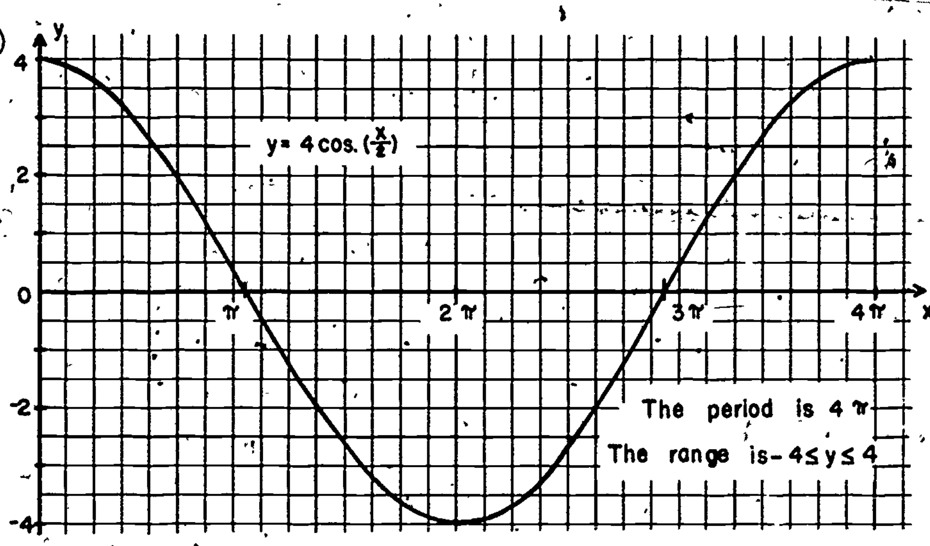
(b)



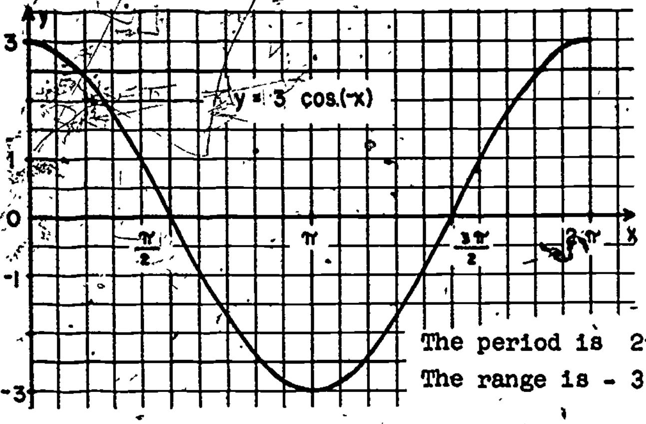
The period is π.

The range is $-3 \leq y \leq 3$.

(c)

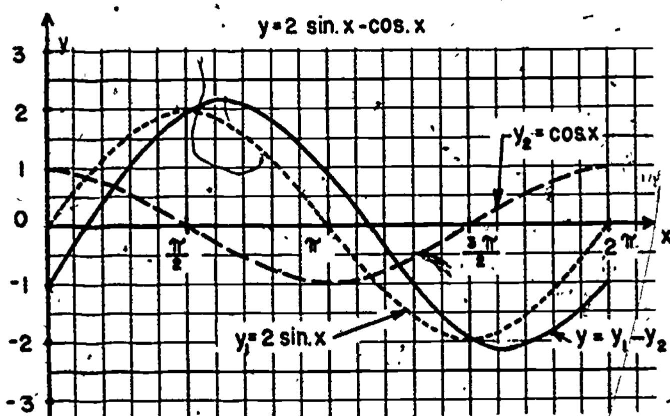


(d)



The period is 2π.
The range is $-3 \leq y \leq 3$.

(e)



The period is 2π. The range is $-\sqrt{5} \leq y \leq \sqrt{5}$.
(We could only expect the student to approximate this range.)

2. Since $x = (\frac{s}{r})t = \omega t$, $st = r\omega t$

(a) If $\omega = \frac{2\pi}{3}$, $r = 3$, then $st = 3 \cdot \frac{2\pi}{3} \cdot t = 2\pi t$

(i) $t = 4$: arc length $= 2\pi(4) = 8\pi$

(ii) $t = 2$: arc length $= 2\pi(2) = 4\pi$

(iii) $t = 6$: arc length $= 2\pi(6) = 12\pi$

(iv) $t = t_0$: arc length $= 2\pi(t_0) = 2\pi t_0$

(b) doubled, tripled

(c) If $\omega = \frac{3\pi}{4}$, $t = 3$, then $st = r \cdot \frac{3\pi}{4} \cdot 3 = \frac{9\pi}{4} \cdot r$

(i) $r = 5$: arc length $= \frac{9\pi}{4} \cdot 5 = \frac{45}{4}\pi$.

(ii) $r = \frac{5}{2}$: arc length $= \frac{9\pi}{4} \cdot \frac{5}{2} = \frac{45}{8}\pi$

(iii) $r = 10$: arc length $= \frac{9\pi}{4} \cdot 10 = \frac{45}{2}\pi$

(iv) $r = R$: arc length $= \frac{9\pi}{4} \cdot R = \frac{9\pi}{4} R$

(d) halved, doubled

(e) If $r = 10$, $t = \frac{9}{2}$, then $st = 10 \cdot \omega \cdot \frac{9}{2} \cdot r$

(i) $\omega = \frac{\pi}{6}$: arc length: $45(\frac{\pi}{6}) = \frac{15}{2}\pi$

(ii) $\omega = \frac{\pi}{3}$: arc length: $45(\frac{\pi}{3}) = 15\pi$

(iii) $\omega = \frac{2\pi}{3}$: arc length: $45(\frac{2\pi}{3}) = 30\pi$

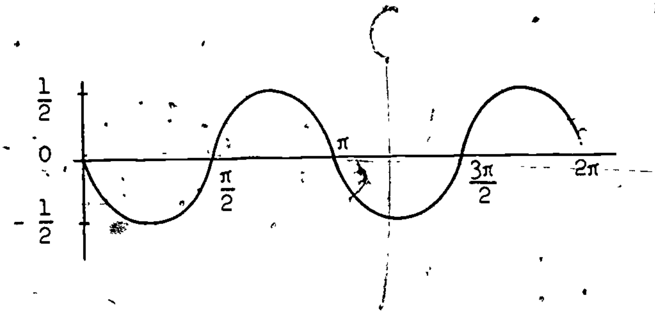
(iv) $\omega = \varphi$: arc length: $45(\varphi) = 45\varphi$

(f) doubled, quadrupled

3. (a) (i) π

(ii) $(\frac{3\pi}{4}, \frac{1}{2})(\frac{7\pi}{4}, \frac{1}{2})$

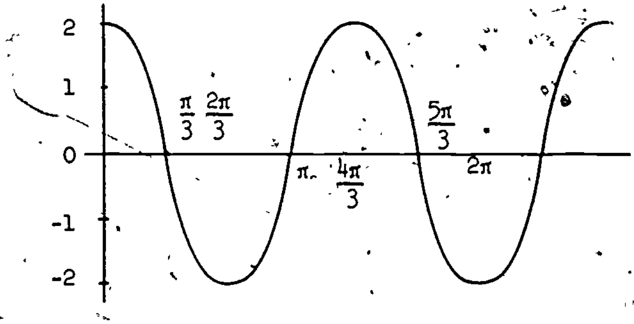
(iii) $(\frac{\pi}{4}, -\frac{1}{2})(\frac{5}{4}\pi, -\frac{1}{2})$



(b) (i) $\frac{4}{3}\pi$

(ii) $(0,2)$, $(\frac{4}{3}\pi,2)$

(iii) $(\frac{2}{3}\pi,-2)$, $(2\pi,-2)$



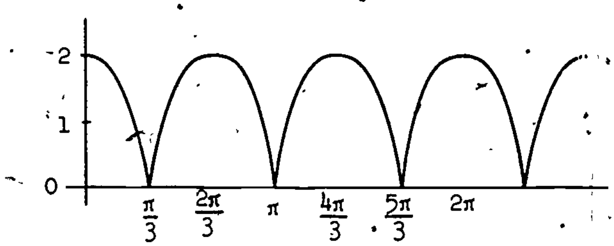
(c) (i) $\frac{2}{3}\pi$

(ii) $(0,2)$, $(\frac{2}{3}\pi,2)$,

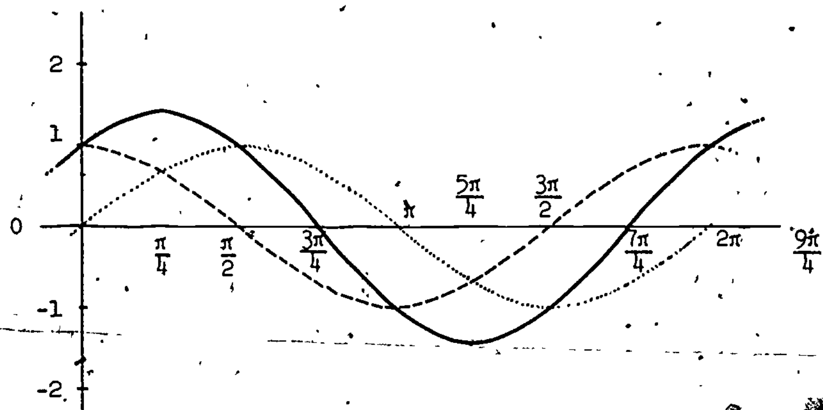
$(\frac{4}{3}\pi,2)$, $(2\pi,2)$.

(iii) $(\frac{\pi}{3},0)$, $(\pi,0)$

$(\frac{5\pi}{3},0)$



4.



(a) (i) .... $y = \sin x$ (see graph)

(ii) -------- $y = \cos x$ (see graph)

(b) (i) ________ $y = \sin x + \cos x$ (see graph)

(ii) The problem anticipates later sections in this chapter. The period is 2π;

the max., $(\frac{\pi}{4}, \sqrt{2})$; and the

min., $(\frac{5\pi}{4}, -\sqrt{2})$.

(c) This graph should be the same. Later, the student will be able to see that

$$\sin x + \cos x = \sqrt{2}(\cos x \cos \frac{\pi}{4} + \sin x \sin \frac{\pi}{4})$$

$$= \sqrt{2} \cos(x - \frac{\pi}{4})$$

and

$$\sin x + \cos x = \sqrt{2}(\sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4})$$

$$= \sqrt{2} \sin(x + \frac{\pi}{4})$$

The student already knows that

$$\sqrt{2} \sin(x + \frac{\pi}{4}) = \sqrt{2} \cos[\frac{\pi}{2} - (x + \frac{\pi}{4})] = \sqrt{2} \cos(\frac{\pi}{4} - x) = \sqrt{2} \cos(x - \frac{\pi}{4})$$

(d) The graphs of (c) and (d) are the same.

Solutions Exercises 3-5

1. (a) In (6) let $\alpha = \beta = x$

$\sin 2x = \sin(x + x) = \sin x \cos x + \cos x \sin x = 2 \sin x \cos x$

(b) $\cos 2x = \cos^2 x - \sin^2 x = 2 \cos^2 x - 1$ (See Example 3-5a)

$= 1 - \sin^2 x - \sin^2 x = 1 - 2 \sin^2 x$

(c) Similar to Example 3-5a, solve for $\sin^2 x$ in (b) above.

Since $\cos 2x = 1 - 2 \sin^2 x$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

2\. (a) Either $\cos x + \sin x = \sqrt{2}(\sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4})$

$$= \underline{\sqrt{2} \sin(x + \tfrac{\pi}{4})}$$

$$= \sqrt{2} \cos(\tfrac{\pi}{2} - (x + \tfrac{\pi}{4}))$$

$$= \sqrt{2} \cos(\tfrac{\pi}{4} - x) = \underline{\sqrt{2} \cos(x - \tfrac{\pi}{4})}$$

or

$$= \sqrt{2}(\cos 2\pi + x - \tfrac{\pi}{4}) = \underline{\sqrt{2} \cos(x + \tfrac{7\pi}{4})}$$

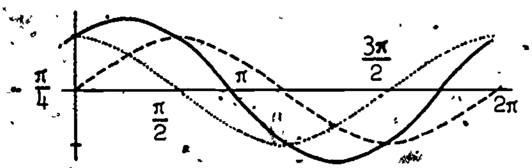
$$\cos x + \sin x = \sqrt{2}(\cos x \cos \tfrac{7\pi}{4} + \sin x(-\sin \tfrac{7\pi}{4}))$$

$$= \underline{\sqrt{2} \cos(x + \tfrac{7\pi}{4})}$$

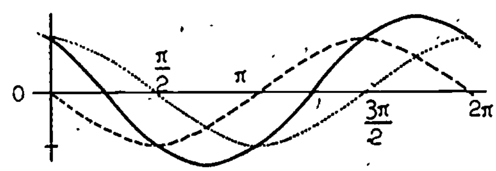
(b) $\cos x - \sin x = \sqrt{2}(\cos x \cos \frac{\pi}{4} - \sin x \sin \frac{\pi}{4})$

$$= \underline{\sqrt{2} \cos(x + \tfrac{\pi}{4})} = \sqrt{2} \sin(\tfrac{\pi}{4} - x)$$

$$= \underline{-\sqrt{2} \sin(x - \tfrac{\pi}{4})} = \sqrt{2} \sin(\pi + x - \tfrac{\pi}{4}) = \underline{\sqrt{2} \sin(x + \tfrac{3\pi}{4})}$$



(a)



(b)

——— $y = \sqrt{2} \cos(x - \frac{\pi}{4})$

$= \sqrt{2} \sin(x + \frac{\pi}{4})$

------- $y = \sin x$

...... $y = \cos x$

——— $y = \sqrt{2} \cos(x + \frac{\pi}{4})$

$= -\sqrt{2} \sin(x - \frac{\pi}{4})$

----- $y = -\sin x$

..... $y = \cos x$

3. (a) $\tan(\alpha - \beta) = \frac{\sin(\alpha - \beta)}{\cos(\alpha - \beta)} = \frac{\sin \alpha \cos \beta - \cos \alpha \sin \beta}{\cos \alpha \cos \beta + \sin \alpha \sin \beta}$

$$= \left(\frac{\frac{1}{\cos \alpha \cos \beta}}{\frac{1}{\cos \alpha \cos \beta}}\right) \cdot \frac{\sin \alpha \cos \beta - \cos \alpha \sin \beta}{\cos \alpha \cos \beta + \sin \alpha \sin \beta} = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}$$

(b) Similar proof to (a) with signs in numerator and denominator interchanged.

(c) In (b) let $\beta = \alpha$.

Then $\tan 2\alpha = \tan(\alpha + \alpha) = \frac{\tan \alpha + \tan \alpha}{1 - \tan \alpha \tan \alpha} = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}$

(d) $\tan \frac{\alpha}{2} = \frac{\sin \frac{\alpha}{2}}{\cos \frac{\alpha}{2}} = \frac{\pm\sqrt{\frac{1 - \cos \alpha}{2}}}{\pm\sqrt{\frac{1 + \cos \alpha}{2}}}$ Using Exercise 1(c)

Using Example 3-5c

$$= \pm\sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha}}$$

$$= \pm\sqrt{\frac{1 - \cos \alpha}{1 + \cos \alpha} \cdot \frac{1 - \cos \alpha}{1 - \cos \alpha}} = \frac{1 - \cos \alpha}{\sin \alpha}$$

$$= \frac{1 - \cos \alpha}{\sin \alpha} \cdot \frac{1 + \cos \alpha}{1 + \cos \alpha} = \frac{\alpha \sin^2 \alpha}{\sin \alpha(1 + \cos \alpha)} = \frac{\sin \alpha}{1 + \cos \alpha}$$

Note: One might think that the result should be $\pm \left(\frac{1 - \cos \alpha}{\sin \alpha}\right)$.

However, if we test quadrant by quadrant we find that $\tan \frac{\alpha}{2}$ and $\frac{1 - \cos \alpha}{\sin \alpha}$ always have the same sign.

If of interest to verify the result

$$\tan \frac{\alpha}{2} = \frac{\sin \alpha}{1 + \cos \alpha}$$

from the accompanying figure A.

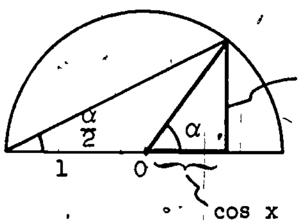


Figure A

Similarly from figure B

$$\tan\frac{\alpha}{2} = \frac{1 - \cos\alpha}{\sin\alpha}.$$

These "proofs" are restricted to the case where $0 < \alpha < \pi$.

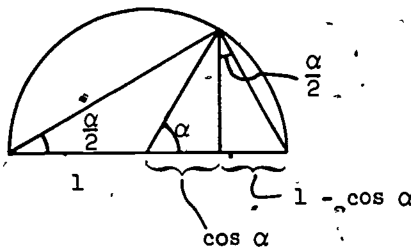
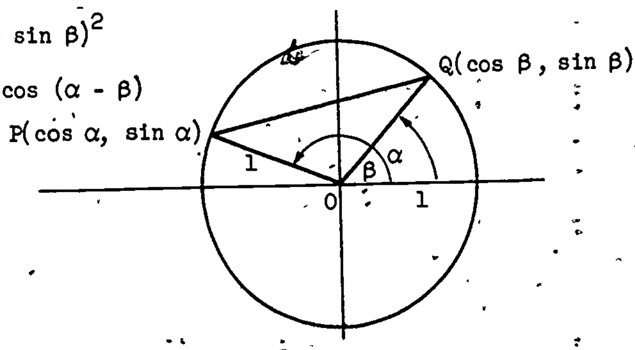


Figure B

4. $(PQ)^2 = (OP)^2 + (OQ)^2 - 2(OP)(OP)\cos(\alpha - \beta)$

$(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$

$= 1^2 + 1^2 - 2(1)(1)\cos(\alpha - \beta)$



$\cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta$

$= 2 - 2\cos(\alpha - \beta)$

$2 - 2(\cos\alpha\cos\beta + \sin\alpha\sin\beta) = 2 - 2\cos(\alpha - \beta)$

$\therefore \cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta$

5. (a) $\frac{1}{2}[\cos(a - b)x - \cos(a + b)x] = \frac{1}{2}[(\cos ax \cos bx + \sin ax \sin bx) - (\cos ax \cos bx - \sin ax \sin bx)]$

$= \sin ax \sin bx$

(b) $\frac{1}{2}[\cos(a - b)x + \cos(a + b)x] = \frac{1}{2}[(\cos ax \cos bx + \sin ax \sin bx) + (\cos ax \cos bx - \sin ax \sin bx)]$

$= \cos ax \cos bx$

6. (a) $\sin \frac{\pi}{12} = \sin(\frac{\pi}{4} - \frac{\pi}{6}) = \sin \frac{\pi}{4} \cos \frac{\pi}{6} - \cos \frac{\pi}{4} \sin \frac{\pi}{6}$

$= (\frac{\sqrt{2}}{2})(\frac{\sqrt{3}}{2}) - (\frac{\sqrt{2}}{2})(\frac{1}{2}) = \frac{\sqrt{6} - \sqrt{2}}{4}$.

Alternatively we can use $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$ derived in Number 1(c). Let $x = \frac{\pi}{12}$; then $2x = \frac{\pi}{6}$ and

$\sin \frac{\pi}{12} = \frac{1}{2}\sqrt{2 - \sqrt{3}} = \frac{\sqrt{6} - \sqrt{2}}{4}$

(b) $\cos \frac{5\pi}{12} = \cos(\frac{\pi}{4} + \frac{\pi}{6}) = \cos \frac{\pi}{4} \cos \frac{\pi}{6} - \sin \frac{\pi}{4} \sin \frac{\pi}{6}$

$= (\frac{\sqrt{2}}{2})(\frac{\sqrt{3}}{2}) - (\frac{\sqrt{2}}{2})(\frac{1}{2}) = \frac{\sqrt{6} - \sqrt{2}}{4}$.

or

$\cos \frac{5\pi}{12} = \sin(\frac{\pi}{2} - \frac{5\pi}{12}) = \sin \frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$.

(c) $\tan \frac{7\pi}{12} = \tan(\frac{\pi}{4} + \frac{\pi}{3}) = \dfrac{\tan \frac{\pi}{4} + \tan \frac{\pi}{3}}{1 - \tan \frac{\pi}{4} \tan \frac{\pi}{3}}$

$= \frac{1 + \sqrt{3}}{1 - \sqrt{3}} = -(2 + \sqrt{3})$

(d) $\cos \frac{11\pi}{12} = \cos(\frac{3\pi}{4} + \frac{\pi}{6})$

$= \cos \frac{3\pi}{4} \cos \frac{\pi}{6} - \sin \frac{3\pi}{4} \sin \frac{\pi}{6}$

$= (-\frac{\sqrt{2}}{2})(\frac{\sqrt{3}}{2}) - (\frac{\sqrt{2}}{2})(\frac{1}{2}) = -(\frac{\sqrt{6} + \sqrt{2}}{4})$

7. (a) $\cos^4\theta - \sin^4\theta = (\cos^2\theta - \sin^2\theta)(\cos^2\theta + \sin^2\theta)$

$$= \cos^2\theta - \sin^2\theta$$

$$= \cos 2\theta$$

(b) $\cos^2\frac{1}{2}\theta = \frac{1}{2}(1 + \cos\theta)$

$$= \frac{\tan\theta + \cos\theta \cdot \tan\theta}{2\tan\theta} \qquad (\tan\theta \neq 0)$$

$$= \frac{\tan\theta + \sin\theta}{2\tan\theta}$$

(c) $(\sin\frac{1}{2}\alpha + \cos\frac{1}{2}\alpha)^2 = \sin^2\frac{1}{2}\alpha + 2\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha + \cos^2\frac{1}{2}\alpha$

$$= 1 + 2\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha$$

$$= 1 + \sin(2\cdot\frac{1}{2})\alpha$$

$$= 1 + \sin\alpha$$

(d) $(\sin\theta + \cos\theta)^2 = \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta$

$$= 1 + 2\sin\theta\cos\theta$$

$$= 1 + \sin 2\theta$$

(e) $\dfrac{2\tan\theta}{1 + \tan^2\theta} = \dfrac{\dfrac{2\sin\theta}{\cos\theta}}{1 + \dfrac{\sin^2\theta}{\cos^2\theta}} = \dfrac{2\sin\theta\cos\theta}{\cos^2\theta + \sin^2\theta}$

$$= 2\sin\theta\cos\theta = \sin 2\theta$$

(f) $\dfrac{1 + \cos\theta}{\sin\theta} + \dfrac{\sin\theta}{1 + \cos\theta} = \dfrac{(1 + \cos\theta)^2 + \sin^2\theta}{\sin\theta\,(1 + \cos\theta)}$

$$= \frac{1 + 2\cos\theta + \cos^2\theta + \sin^2\theta}{\sin\theta\,(1 + \cos\theta)}$$

$$= \frac{2(1 + \cos\theta)}{\sin\theta(1 + \cos\theta)}$$

$$= \frac{2}{\sin\theta}.$$

[Neither side of the identity is defined if $\theta = n\pi$.]

8. (a) $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

(b) $\cos^4 x = \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$

$$= \frac{1}{4}[1 + 2\cos 2x + \frac{1}{2}(1 + \cos 4x)]$$

$$= \frac{1}{8}[3 + 4\cos 2x + \cos 4x]$$

9. $\cos 2x = 1 - 2\sin^2 x$

$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$

$\sin^4 x = \frac{1}{4}(1 - 2\cos 2x + \cos^2 2x)$

$$= \frac{1}{4}[1 - 2\cos 2x + \frac{1}{2}(1 + \cos 4x)]$$

$$= \frac{1}{8}[3 - 4\cos 2x + \cos 4x]$$

10. (a) $\sin 2\theta \cos\theta - \cos 2\theta \sin\theta = \sin(2\theta - \theta) = \sin\theta$

(b) $\sin(x - y)\cos z + \sin(y - z)\cos x = (\sin x \cos y - \cos x \sin y)\cos z$

$$+ (\sin y \cos z - \cos y \sin z)\cos x$$

$$= \sin x \cos y \cos z - \cos y \sin z \cos x$$

$$= (\sin x \cos z - \sin z \cos x)\cos y$$

$$= \sin(x - z)\cos y$$

(c) $\sin 3x \sin 2x = \frac{1}{2}[\cos(3x - 2x) - \cos(3x + 2x)]$

$$= \frac{1}{2}(\cos x - \cos 5x)$$

(d) $\cos\theta - \sin\theta \tan 2\theta = \cos\theta - \sin\theta \frac{\sin 2\theta}{\cos 2\theta}$

$$= \frac{\cos\theta \cos 2\theta - \sin\theta \sin 2\theta}{\cos 2\theta}$$

$$= \frac{\cos(2\theta + \theta)}{\cos 2\theta} = \frac{\cos 3\theta}{\cos 2\theta}$$ (Valid if $\cos 2\theta \neq 0$.)

(e) $\sin 3\theta = \sin(2\theta + \theta) = \sin 2\theta \cos \theta + \cos 2\theta \sin \theta$

$$= 2 \sin \theta \cos^2 \theta + (1 - 2 \sin^2 \theta)\sin \theta$$

$$= 2 \sin \theta(1 - \sin^2 \theta) + \sin \theta - 2 \sin^3 \theta$$

$$= 2 \sin \theta - 2 \sin^3 \theta + \sin \theta - 2 \sin^3 \theta$$

$$= 3 \sin \theta - 4 \sin^3 \theta$$

Hence $\sin^3 \theta = \frac{1}{4}(3 \sin \theta - \sin 3\theta)$.

(f) $\sin 3x + \sin x = 2 \sin 2x \cos x$

Hence, $\sin x + \sin 2x + \sin 3x = \sin 2x[1 + 2 \cos x]$

(g) $$\frac{1 + \tan x}{1 - \tan x} = \frac{1 + \frac{\sin x}{\cos x}}{1 - \frac{\sin x}{\cos x}} = \frac{\cos x + \sin x}{\cos x - \sin x},$$

Hence

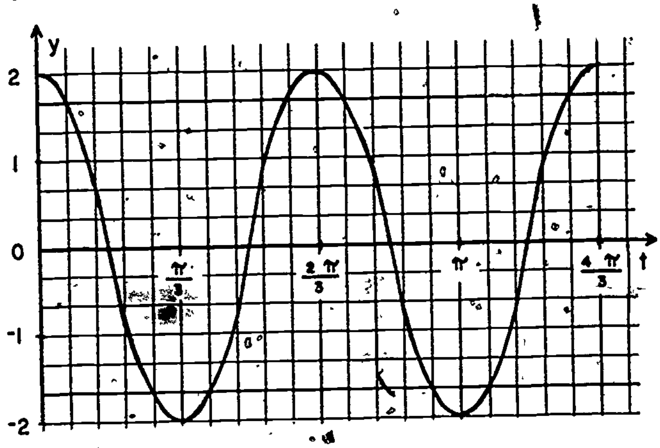
$$\left(\frac{1 + \tan x}{1 - \tan x}\right)^2 = \frac{\cos^2 x + 2 \sin x \cos x + \sin^2 x}{\cos^2 x - 2 \sin x \cos x + \cos^2 x}$$

$$= \frac{1 + 2 \sin x \cos x}{1 - 2 \sin x \cos x}$$

$$= \frac{1 + \sin 2x}{1 - \sin 2x}$$

Solutions Exercises 3-6

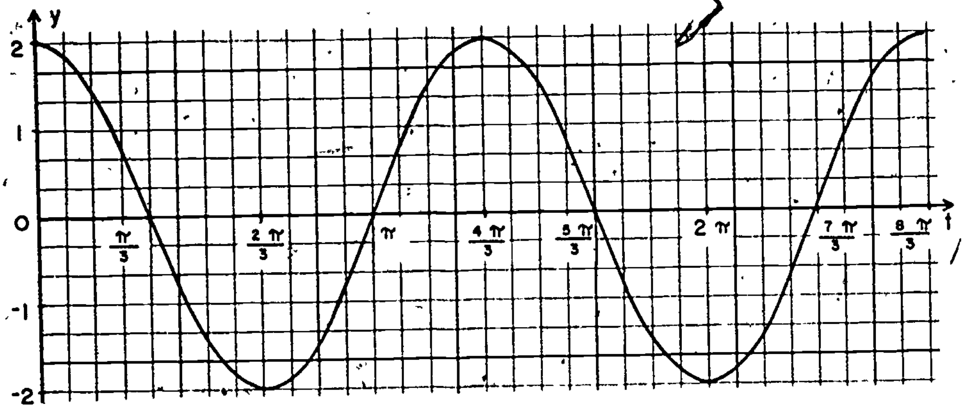
1. (a)



Amplitude = 2,

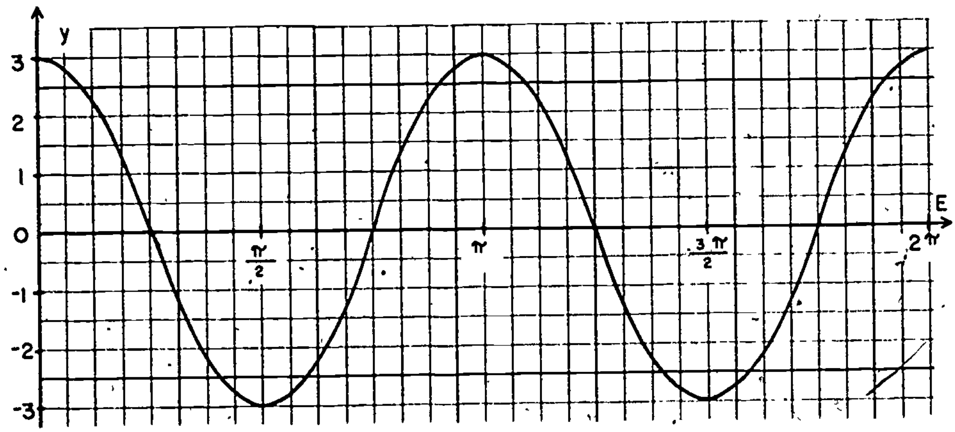
Period = $\frac{2\pi}{3}$.

(b)



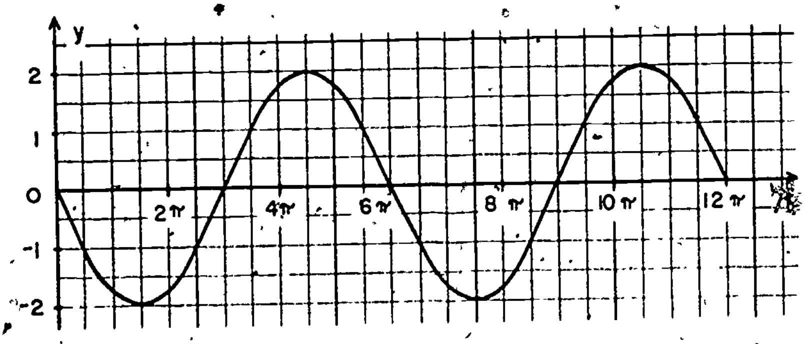
Amplitude = 2, Period = $\frac{4\pi}{3}$

(c)



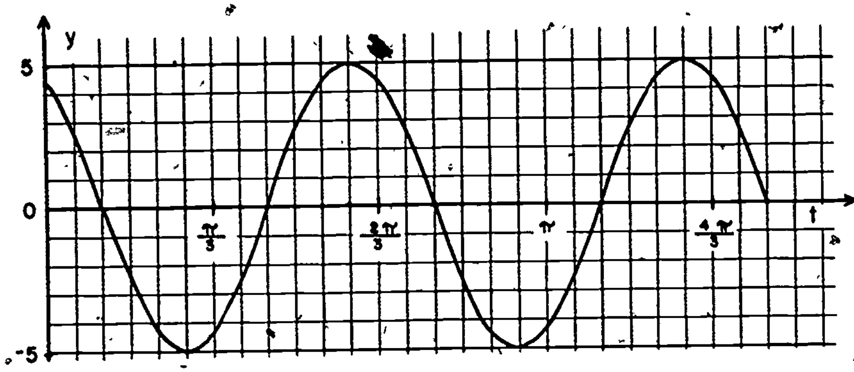
Amplitude = 3, Period = π

(d)



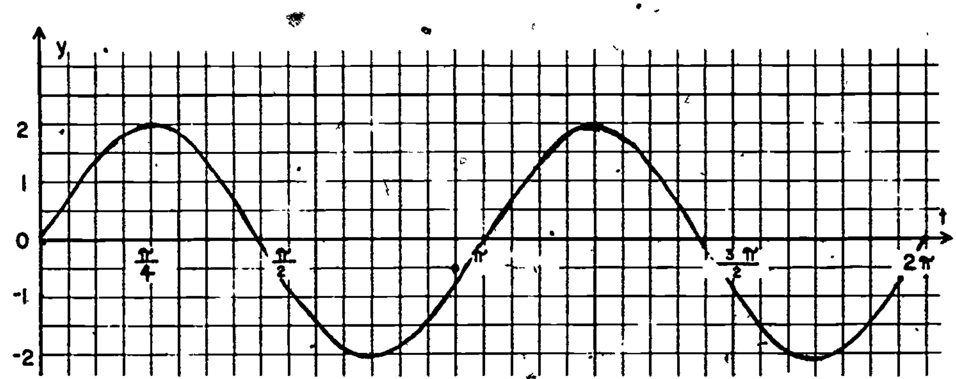
Amplitude = 2, Period = 6π

(e)



Amplitude = 2, Period = π.

(f)



Amplitude = 5, Period = $\frac{2\pi}{3}$.

2.

	Amplitude	Period		Amplitude	Period
(a)	$\sqrt{2}$	$\frac{2\pi}{3}$	(d)	10	6
(b)	$\sqrt{5}$	2	(e)	$\frac{13}{2}$	5π
(c)	$2\sqrt{2}$	$\frac{4\pi}{3}$	(f)	$3\sqrt{3}$	$\frac{8}{5}$

3. (a). $y = \sin x + \sqrt{3}\cos x$

$$= 2[\sin x \cdot \frac{1}{2} + \cos x \cdot \frac{\sqrt{3}}{2}]$$

$$= 2(\sin x \cos \frac{\pi}{3} + \cos x \sin \frac{\pi}{3})$$

$$= 2 \sin (x + \frac{\pi}{3})$$

(b) $y = -\sin x + \cos x$

$$= \sqrt{2}\ [\sin x(\frac{-1}{\sqrt{2}}) + \cos x\ (\frac{1}{\sqrt{2}})]$$

$$= \sqrt{2}\ (\sin x \cos \frac{3\pi}{4} + \cos x \sin \frac{3\pi}{4})$$

$$= \sqrt{2} \sin (x + \frac{3\pi}{4})$$

(c) $y = -\sqrt{3}\sin x - \cos x$

$$= 2[\sin x\ (\frac{-\sqrt{3}}{2}) + \cos x\ (-\frac{1}{2})]$$

$$= 2(\sin x \cos \frac{7\pi}{6} + \cos x \sin \frac{7\pi}{6})$$

$$= 2 \sin(x + \frac{7\pi}{6})$$

(d) $y = \sin x - \cos x$

$$= \sqrt{2}\ [\sin x(\frac{1}{\sqrt{2}}) + \cos x\ (-\frac{1}{\sqrt{2}})]$$

$$= \sqrt{2}(\sin x \cos \frac{7\pi}{4} + \cos x \sin \frac{7\pi}{4})$$

$$= \sqrt{2} \sin(x + \frac{7\pi}{4})$$

4. (a) (i) $y = \sqrt{3}\cos x + \sin x$

$$= 2[\cos x\ (\frac{\sqrt{3}}{2}) - \sin x\ (-\frac{1}{2})]$$

$$= 2(\cos x \cos \frac{11\pi}{6} - \sin x \sin \frac{11\pi}{6})$$

$$= 2 \cos\ (x + \frac{11\pi}{6})$$

(ii) $y = \sin(x + \frac{\pi}{3})$ (Soln. of Exercise 3(a))

$$= \sin(x + \frac{\pi}{3} + 2\pi) = \sin(x + \frac{7\pi}{3})$$

$$= \cos(\frac{\pi}{2} - x - \frac{7\pi}{3}) = \cos(-x - \frac{11\pi}{6})$$

$$= \cos(x + \frac{11\pi}{6})$$

(b) (i) $y = \cos x - \sin x$

$= \sqrt{2}\,[\cos x(\frac{1}{\sqrt{2}}) - \sin x(\frac{1}{\sqrt{2}})]$

$= \sqrt{2}\,(\cos x \cos \frac{\pi}{4} - \sin x \sin \frac{\pi}{4})$

$= \sqrt{2}\,\cos(x + \frac{\pi}{4})$

(ii) $y = \sqrt{2}\,\sin(x + \frac{3\pi}{4})$ (Soln. of Exercise 3(f))

$= \sqrt{2}\,\cos(\frac{\pi}{2} - x - \frac{3\pi}{4})$

$= \sqrt{2}\,\cos(-x - \frac{\pi}{4}) = \sqrt{2}\,\cos(x + \frac{\pi}{4})$

(c) (i) $y = -\cos x - \sqrt{3}\sin x$

$= 2[\cos x(-\frac{1}{2}) - \sin x(\frac{\sqrt{3}}{2})]$

$= 2(\cos x \cos \frac{2\pi}{3} - \sin x \sin \frac{2\pi}{3})$

$= 2\cos(x + \frac{2\pi}{3})$

(ii) $y = 2\sin(x + \frac{7\pi}{6})$ (Soln. of Exercise 3(c))

$= 2\cos(\frac{\pi}{2} - x - \frac{7\pi}{6}) = 2\cos(-x - \frac{2\pi}{3})$

$= 2\cos(x + \frac{2\pi}{3})$

(d) (i) $y = -\cos x + \sin x$

$= \sqrt{2}\,[\cos x(-\frac{1}{\sqrt{2}}) - \sin x(-\frac{1}{\sqrt{2}})]$

$= \sqrt{2}\,(\cos x \cos \frac{5\pi}{4} - \sin x \sin \frac{5\pi}{4})$

$= \sqrt{2}\,\cos(x + \frac{5\pi}{4})$

(ii) $y = \sqrt{2}\,\sin(x + \frac{7\pi}{4})$ (Soln. of Exercise 3(d))

$= \sqrt{2}\,\cos(\frac{\pi}{2} - x - \frac{7\pi}{4})$

$= \sqrt{2}\,\cos(-x - \frac{5\pi}{4}) = \sqrt{2}\,\cos(x + \frac{5\pi}{4})$

5. $y = 9\sqrt{2}\sin \pi t - 3\sqrt{6}\cos \pi t$

(a) $A = \sqrt{81 \cdot 2 + 9 \cdot 6} = 3\sqrt{24} = 6\sqrt{6}$ (amplitude)

$$y = 6\sqrt{6}\left[\frac{9\sqrt{2}}{6\sqrt{6}}\sin \pi t - \frac{3\sqrt{6}}{6\sqrt{6}}\cos \pi t\right]$$

$$= 6\sqrt{6}\left(\frac{\sqrt{3}}{2}\sin \pi t - \frac{1}{2}\cos \pi t\right)$$

(i) Form $y = A \sin (\pi t + \alpha)$:

$$y = 6\sqrt{6}\left[\sin \pi t\left(\frac{\sqrt{3}}{2}\right) + \cos \pi t\left(-\frac{1}{2}\right)\right]$$

$$= 6\sqrt{6}\left(\sin \pi t \cos \frac{11\pi}{6} + \cos \pi t \sin \frac{11\pi}{6}\right)$$

$$= 6\sqrt{6}\sin\left(\pi t + \frac{11\pi}{6}\right)$$

(ii) Form $y = A \sin (\pi t - \alpha)$:

$$y = 6\sqrt{6}\left[\sin \pi t\left(\frac{\sqrt{3}}{2}\right) - \cos \pi t\left(\frac{1}{2}\right)\right]$$

$$= 6\sqrt{6}\left(\sin \pi t \cos \frac{\pi}{6} - \cos \pi t \sin \frac{\pi}{6}\right)$$

$$= 6\sqrt{6}\sin\left(\pi t - \frac{\pi}{6}\right)$$

(iii) Form $y = A \cos (\pi t - \alpha)$:

$$= 6\sqrt{6}\left[\left(\cos \pi t\left(-\frac{1}{2}\right) + \sin \pi t\left(\frac{\sqrt{3}}{2}\right)\right)\right.$$

$$= 6\sqrt{6}\left(\cos \pi t \cos \frac{2\pi}{3} + \sin \pi t \sin \frac{2\pi}{3}\right)$$

$$= 6\sqrt{6}\left(\cos \pi t - \frac{2\pi}{3}\right)$$

(iv) Form $y = A \cos (\pi t + \alpha)$:

$$= 6\sqrt{6}\left[\cos \pi t\left(-\frac{1}{2}\right) + \sin \pi t\left(-\frac{\sqrt{3}}{2}\right)\right]$$

$$= 6\sqrt{6}\left(\cos \pi t \cos \frac{4\pi}{3} - \sin \pi t \sin \frac{4\pi}{3}\right)$$

$$= 6\sqrt{6}\cos\left(\pi t + \frac{4\pi}{3}\right)$$

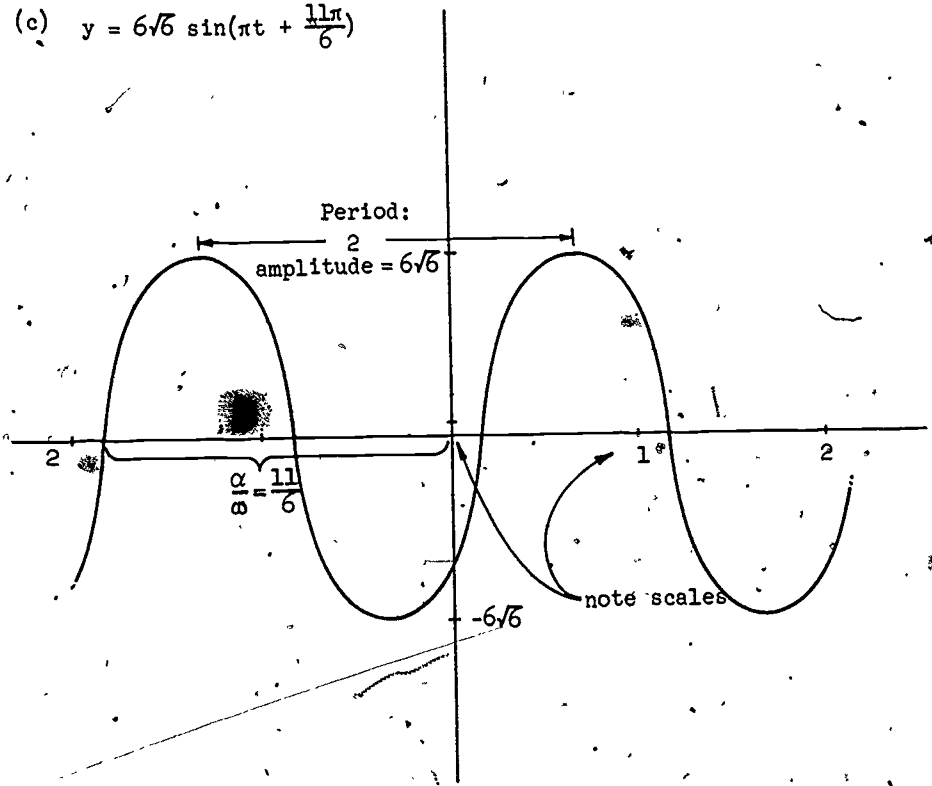
(b) $\sin(\pi t - \frac{\pi}{6})$ (ii)

$= \sin(\pi t - \frac{\pi}{6} + 2\pi) = \sin(\pi t + \frac{11\pi}{6})$ (i)

$= \cos(\frac{\pi}{2} - \pi t - \frac{11\pi}{6}) = \cos(-\pi t - \frac{4\pi}{3}) = \cos(\pi t + \frac{4\pi}{3})$ (iv)

$= \cos(\pi t + \frac{4\pi}{3} - 2\pi) = \cos(\pi t - \frac{2\pi}{3})$ (iii)

(c) $y = 6\sqrt{6}\,\sin(\pi t + \frac{11\pi}{6})$



6. (a) $y = A\cos(\omega t - \alpha) = A\cos\omega t\cos\alpha + A\sin\omega t\sin\alpha$

$y = 4\sin\pi t - 3\cos\pi t$

$A\sin\alpha = 4,\quad A\cos\alpha = -3$

$A^2(\sin^2\alpha + \cos^2\alpha) = 16 + 9$

$A^2 = 25,\quad A = 5$

$\sin\alpha = \frac{4}{5},\quad \cos\alpha = -\frac{3}{5},$

$\alpha \approx \pi - 0.927 \approx 2.215$

Answer: $y = 5\cos(\pi t - 2.215)$

(b) $y = -4 \sin \pi t + 3 \cos \pi t$

$A \sin \alpha = -4, \quad A \cos \alpha = 3, \quad A = 5.$

$\sin \alpha = -\frac{4}{5}, \quad \cos \alpha = \frac{3}{5},$

$a \approx 2\pi - 0.927 \approx 5.357$

Answer: $y = 5 \cos (\pi t - 5.357).$

(c) $y = -4 \sin \pi t - 3 \cos \pi t$

$A = 5, \quad \sin \alpha = -\frac{4}{5}, \quad \cos \alpha = -\frac{3}{5}.$

$a \approx \pi + 0.927 \approx 4.069$

Answer: $y = 5 \cos (\pi t - 4.069)$

(d) $y = 3 \sin \pi t + 4 \cos \pi t$

$A = 5, \quad \sin \alpha = \frac{3}{5}, \quad \cos \alpha = \frac{4}{5}, \quad \alpha \approx 0.644$

Answer: $y = 5 \cos (\pi t - 0.644)$

(e) $y = 3 \sin \pi t - 4 \cos \pi t$

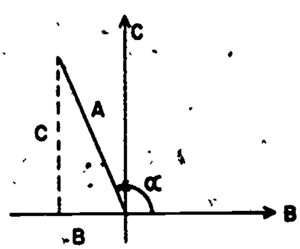
$A = 5, \quad \sin \alpha = \frac{3}{5}, \quad \cos \alpha = -\frac{4}{5}$

$\alpha \approx \pi - 0.644 \approx 2.498$

Answer: $y = 5 \cos (\pi t - 2.498).$

7. $A^2 = B^2 + C^2$

$\sin \alpha = \frac{C}{A}, \quad \cos \alpha = \frac{B}{A}.$



*Although the directions in this problem do not ask for the values of t at which the maxima and minima occur, they have been included in these solutions in case the question arises.

(a) $A = 5$, $\sin\alpha = \frac{3}{5}$, $\cos\alpha = \frac{4}{5}$, $\alpha \approx 0.644$.

Hence, $3\sin 2t + 4\cos 2t \approx 5\cos(2t - 0.644)$.

Maximum value, 5, occurs when $\cos(2t - 0.644) = 1$, or

$$2t - 0.644 = 0, \quad t = 0.322.$$

Minimum value, -5, occurs when $\cos(2t - 0.644) = -1$, or

$$2t - 0.644 = \pi, \quad t \approx 1.893.$$

The period $= \frac{2\pi}{\omega} = \frac{2\pi}{2} = \pi$.

Hence, maximum values occur at $t \approx 0.322 + n\pi$ and minimum values at $t \approx 1.893 + n\pi$.

(b) $A = \sqrt{4 + 9} = \sqrt{13}$, $\sin\alpha = \frac{2}{\sqrt{13}}$, $\cos\alpha = \frac{-3}{\sqrt{13}}$,

$\alpha \approx \pi - 0.589 \approx 2.553$.

Hence, $2\sin 3t - 3\cos 3t \approx \sqrt{13}\cos(3t - 2.553)$.

The period $= \frac{2\pi}{3}$. Maximum values, $\sqrt{13}$, occur when

$$3t - 2.553 = 0 + 2n\pi, \quad t \approx 0.851 + \frac{2n\pi}{3}.$$

Minimum values, $-\sqrt{13}$, occur when

$$3t - 2.553 = \pi + 2n\pi, \quad t \approx 1.898 + \frac{2n\pi}{3}.$$

(c) $A = \sqrt{1 + 1} = \sqrt{2}$, $\sin\alpha = -\frac{1}{\sqrt{2}}$, $\cos\alpha = \frac{1}{\sqrt{2}}$, $\alpha = \frac{7\pi}{4}$

Hence, $-\sin(\frac{t}{2}) + \cos(\frac{t}{2}) = \sqrt{2}\cos(\frac{t}{2} - \frac{7\pi}{4})$.

The period $= \frac{2\pi}{\frac{1}{2}} = 4\pi$. Maximum values, $\sqrt{2}$, occur when

$$\frac{t}{2} - \frac{7\pi}{4} = 0 + 2n\pi, \quad t = \frac{7\pi}{2} + 4n\pi.$$

Minimum values, $-\sqrt{2}$ occur when

$$\frac{t}{2} - \frac{7\pi}{4} = \pi + 2n\pi, \quad t = \frac{3\pi}{2} + 4n\pi.$$

8. $A \cos(\omega t - \alpha) + B \cos(\omega t - \beta)$

$$= A \cos \omega t \cos \alpha + A \sin \omega t \sin \alpha + B \cos \omega t \cos \beta + B \sin \omega t \sin \beta$$

$$= (A \cos \alpha + B \cos \beta) \cos \omega t + (A \sin \alpha + B \sin \beta) \sin \omega t$$

$$= C \cos(\omega t - \gamma) \quad \text{when}$$

$$C = \sqrt{(A \sin \alpha + B \sin \beta)^2 + (A \cos \alpha + B \cos \beta)^2},$$

$$\sin \gamma = \frac{A \sin \alpha + B \sin \beta}{C}, \quad \text{and} \quad \cos \gamma = \frac{A \cos \alpha + B \cos \beta}{C}.$$

Since A, B, α, and β are real numbers, it follows that C is a real number, and it is easy to show that

$$0 \leq \sin^2 \gamma \leq 1, \quad 0 \leq \cos^2 \gamma \leq 1, \quad \sin^2 \gamma + \cos^2 \gamma = 1,$$

and therefore γ is a real number.

9. Given $y = B \cos(\mu t - \beta)$.

We may assume that $0 < \beta < 2\pi$.

1. If μ and B are positive, we set $\mu = \omega$, $B = A$, $\beta = \alpha$.
2. If μ is positive and B is negative, set $\mu = \omega$, $B = -A$.

   Then $y = A[-\cos(\omega t - \beta)] = A \cos(\omega t - \beta \pm \pi)$.

   If $0 \leq \beta < \pi$, take $\alpha = \beta + \pi$.

   If $\pi \leq \beta < 2\pi$, take $\alpha = \beta - \pi$.
3. If μ is negative, set $\mu = -\omega$.

   Then $y = B \cos(-\omega t - \beta) = B \cos(\omega t + \beta)$

   $= B \cos[\omega t - (2\pi - \beta)]$

   $= B \cos(\omega t - \beta')$.

Proceed as in 1 and 2.

Solutions Exercises 3-7

1. (a) $m(2\pi) = n(\pi)$

  $\therefore \quad m = 1; \quad n = 2;$ and period is 2π

 (b) $m(4\pi) = n(6\pi)$

  $\therefore \quad m = 3; \quad n = 2;$ and period is 12π

 (c) $m(4) = n(2)$

  $\therefore \quad m = 1; \quad n = 2;$ and period is 4

 (d) $m(\frac{\pi}{6}) = n(\frac{4\pi}{3})$

  $\therefore \quad m = 8; \quad n = 1;$ and period is $\frac{4\pi}{3}$

 (e) $1 - 2\sin^2 x + 2\sin\frac{x}{2}\cos\frac{x}{2} = \cos 2x + \sin x$

  $m(\pi) = n(2\pi)$

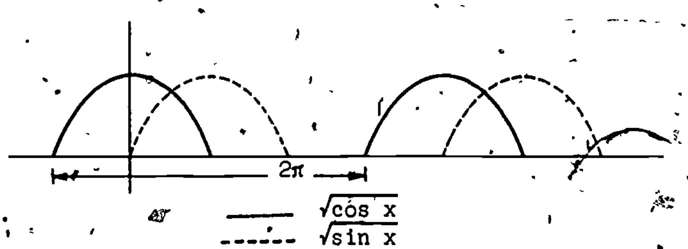
  $m = 2; \quad n = 1$ and period $= 2\pi$

 (f) $m(\pi) = n(\frac{\pi}{2})$

  $m = 1; \quad n = 2;$ and period $= 2\pi$

 (g) $m(2\pi) = n(2\pi)$

  $m = 1; \quad n = 1;$ and period $= 2\pi$



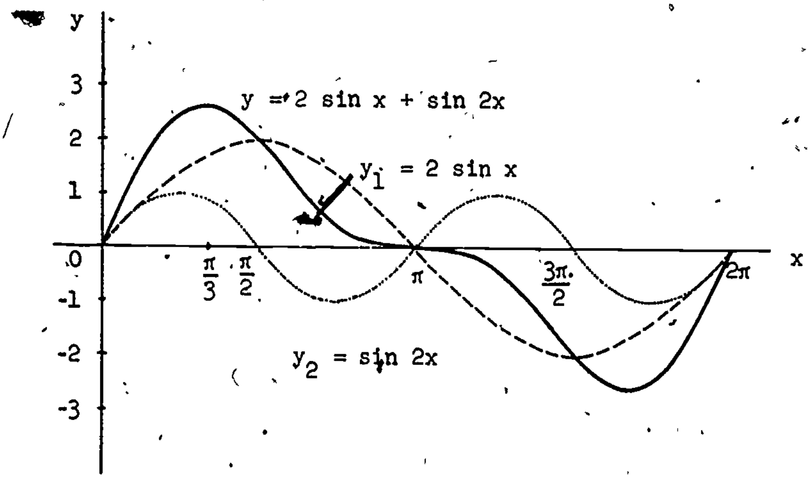
 (h) Period of $\sin\frac{3\pi x}{2}$ is $\frac{4}{3}$; period of $\left|\sin\frac{3\pi x}{2}\right|$ is $\frac{2}{3}$

  Period of $\cos 4\pi x$ is $\frac{1}{2}$; period of $|\cos 4\pi x|$ is $\frac{1}{4}$

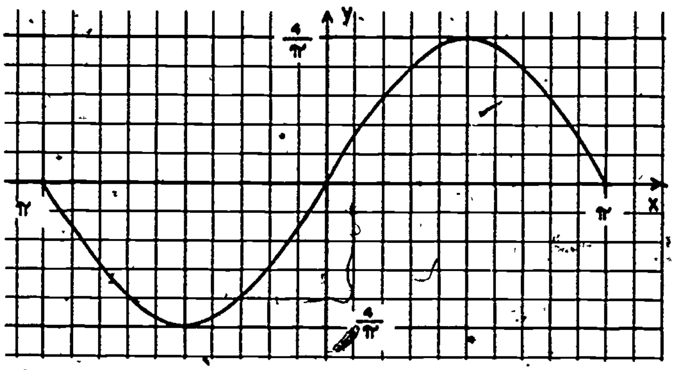
  $\therefore \quad m(\frac{2}{3}) = n(\frac{1}{4})$

  $m = 3; \quad n = 8;$ and period is 2.

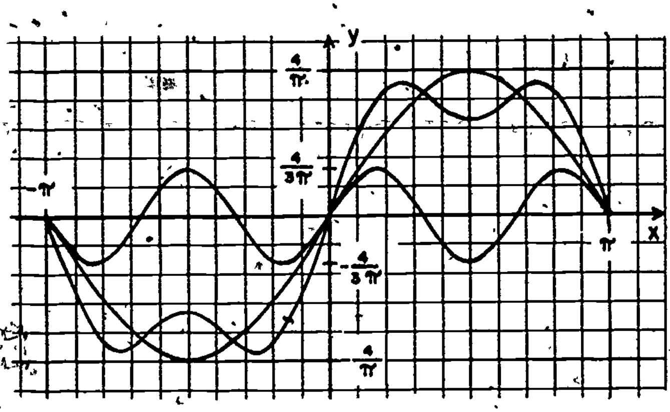
2. The function $2 \sin x$ has period 2π and amplitude 2 while $\sin 2x$ has period π and amplitude 1.



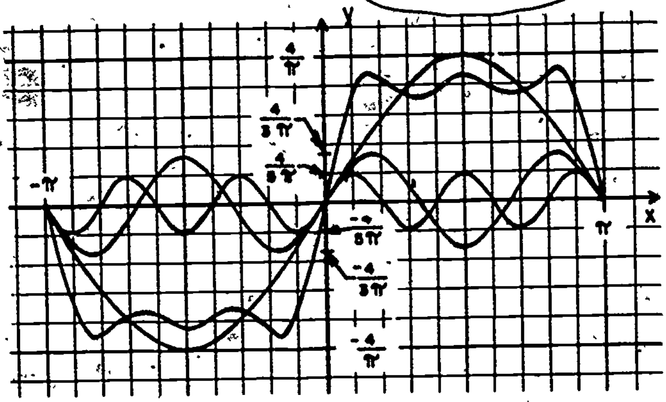
3. (a)



(b)



(c)

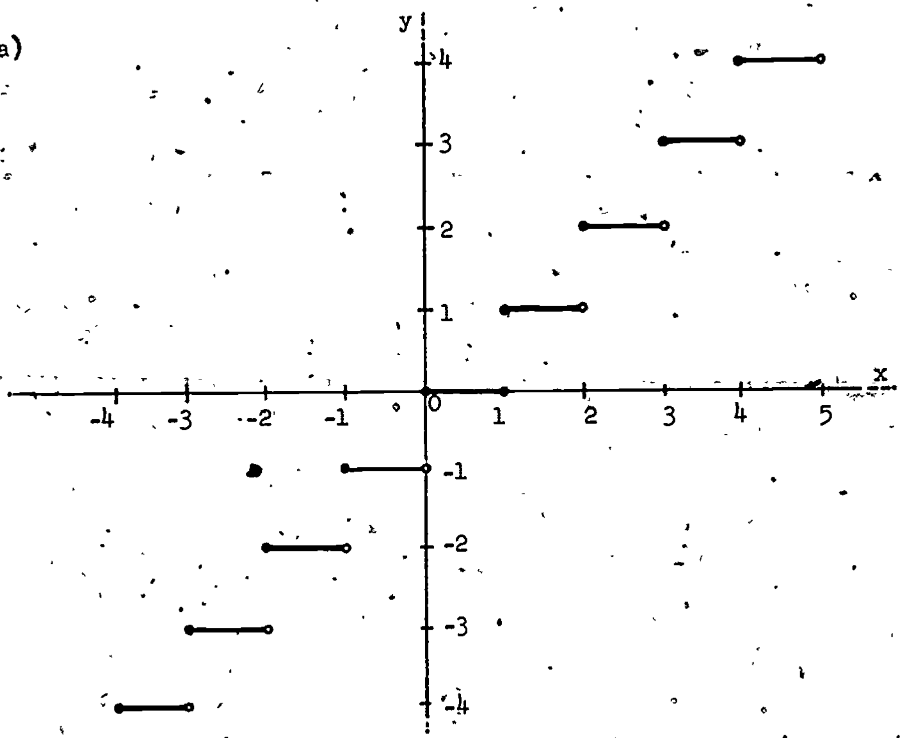


4. (a) 2π, $\frac{2\pi}{3}$, $\frac{2\pi}{5}$, ...

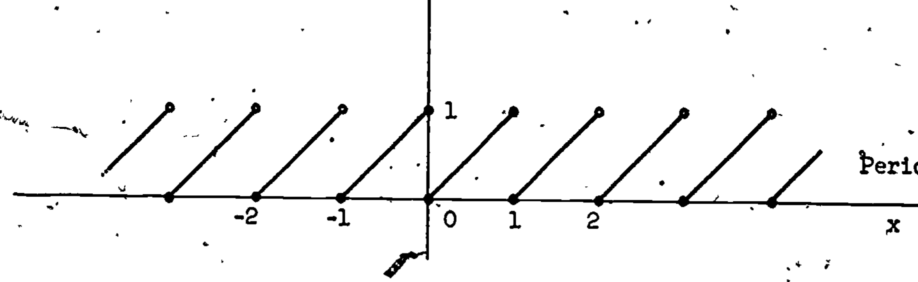
(b) The cosine terms; also the terms $B_n \sin nx$, n even.

(in our case, $a = 2\pi$.) The function being represented has the property that $f(-x) = -f(x)$ [odd function]. This property holds for $\sin nx$ but not $\cos nx$. Moreover, $f(x)$ has the property that, $f(\pi - x) = f(x)$. This property does not hold for $\sin 2kx$, k integral, since $\sin 2k(\pi - x) = -\sin 2kx$. It does hold for $\sin(2k + 1)x$.

5. (a)

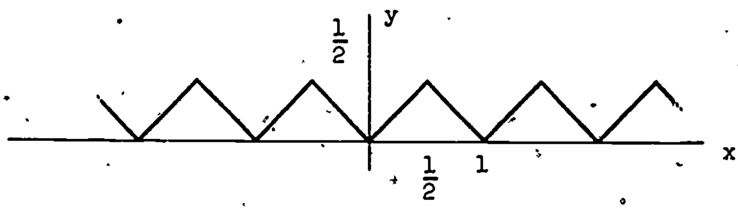


(b) $f(x) = x - [x]$



Period: 1

(c) (i) $y = (x)$



Period: 1

The maximum value is $\frac{1}{2}$; the minimum, 0.

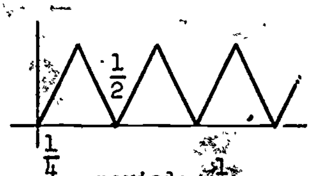
In the interval $[0,1]$ we have

$$(x) = \begin{cases} x & 0 < x < \frac{1}{2} \\ 1 - x & \frac{1}{2} < x \leq 1 \end{cases}$$

Since $(x + 1) = (x)$ for all x, we have a periodic function of x whose period is 1.

(ii)

$y = (2x)$

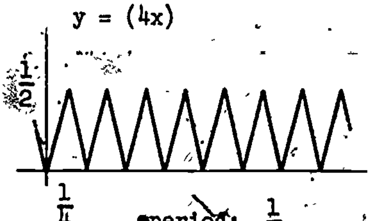


period: $\frac{1}{2}$

max: $\frac{1}{2}$

min: 0

$y = (4x)$



period: $\frac{1}{4}$

max: $\frac{1}{2}$

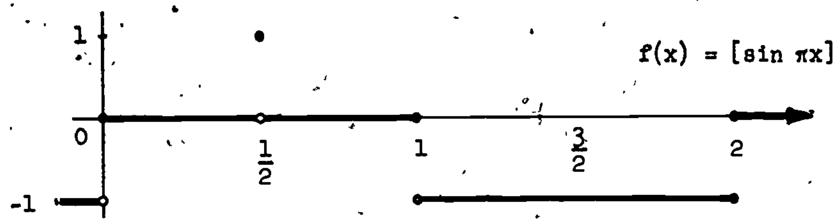
min: 0

(iii) (nx) period: $\frac{1}{n}$

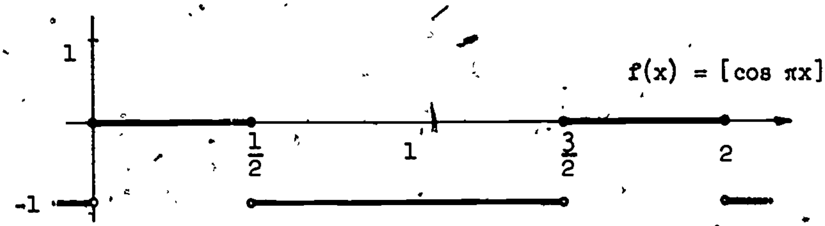
max: $\frac{1}{2}$

min: 0

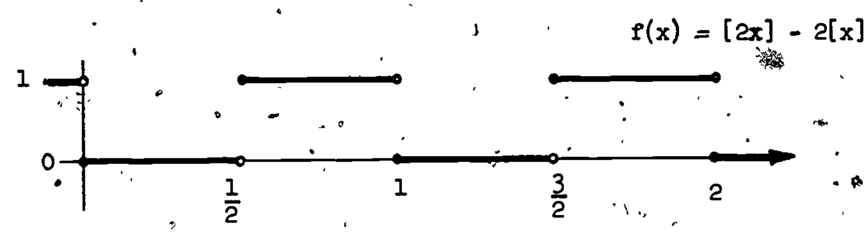
6. (a)



(b)



(c)



7. The function is periodic since

$$f(x + r) = f(x)$$

for any rational number r. The function has no fundamental period, however, because there is no <u>smallest</u> positive rational number r. See Appendix A1-1.

Chapter 4

DERIVATIVES OF CIRCULAR FUNCTIONS

Following the initial general type of argument used in Chapter 2, one can show that near the origin, the graph of $y = \sin x$ lies in the wedge between the line given by $y = x$ and $y = (1 - \epsilon)x$ for any positive ϵ, however small; hence the line tangent at the origin to the graph of $y = \sin x$ is given by $y = x$. We begin the chapter with a similar argument showing that the line tangent at $(0,1)$ to the graph of $y = \cos x$ is the horizontal line given by $y = 1$. We use the specific results, together with the definition of derivative (as the limit of a difference quotient) and addition formula (6) of Section 3-5 to show that $D(\sin x) = \cos x$ in Section 4-2.

We could have generalized the methods of Section 4-1 to an arbitrary point $(a,f(a))$ of the graph by expressing $\sin x$ and $\cos x$ in terms of $x - a$. This would also have led to the result $D \sin x = \cos x$.

The addition formulas of Section 3-5 provide us with a device for finding tangents to the graphs of the circular functions at arbitrary points. Consider first the sine function and the problem of finding the tangent to its graph at the point $(a, \sin a)$. Since

$$x = a + (x - a),$$

we can write

$$\sin x = \sin [a + (x - a)].$$

Using addition formula (6) we get

$$\sin x = \sin a \cos (x - a) + \cos a \sin (x - a). \tag{1}$$

If x is close to a, then $x - a$ is small. Thus we can replace $\cos (x - a)$ and $\sin (x - a)$ in (1) by their best linear approximations, (using (1) and (7) of the last section). We have $\cos(x - a) \approx 1$ and $\sin (x - a) \approx x - a$. Therefore, for x near a, we can write

$$\sin x \approx (\sin a)(1) + (\cos a)(x - a)$$

as the best linear approximation. The equation of the tangent line at the point $(a , \sin a)$ is

$$y = \sin x + (\cos a)(x - a). \tag{2}$$

For example, if $a = \frac{\pi}{4}$, then

$$\sin a = \cos a = \frac{\sqrt{2}}{2}$$

and the tangent to the graph of $y = \sin x$ at $(\frac{\pi}{4}, \sin\frac{\pi}{4})$ has the equation

$$y = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \frac{\pi}{4}).$$

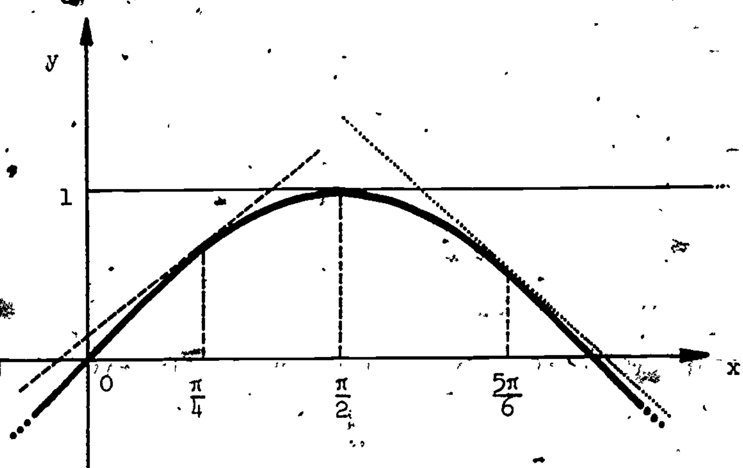
Similarly, the tangent at $a = \frac{\pi}{2}$ has the equation

$$\begin{aligned} y &= \sin\frac{\pi}{2} + (\cos\frac{\pi}{2})(x - \frac{\pi}{2}) \\ &= 1 + 0(x - \frac{\pi}{2}) \\ &= 1; \end{aligned}$$

that is, the tangent is horizontal at the point $(\frac{\pi}{2}, 1)$.

Let us consider another point. The tangent at $(\frac{5\pi}{6}, \frac{1}{2})$ has the equation

$$\begin{aligned} y &= \sin\frac{5\pi}{6} + (\cos\frac{5\pi}{6})(x - \frac{5\pi}{6}) \\ &= \frac{1}{2} - \frac{\sqrt{3}}{2}(x - \frac{5\pi}{6}). \end{aligned}$$



$y = 1$ ———————— the tangent at $(\frac{\pi}{2}, 1)$

$y = \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(x - \frac{\pi}{4})$ ------- the tangent at $(\frac{\pi}{4}, \frac{\sqrt{2}}{2})$

$y = \frac{1}{2} - \frac{\sqrt{3}}{2}(x - \frac{5\pi}{6})$ .... the tangent at $(\frac{5\pi}{6}, \frac{1}{2})$

All of our illustrative equations for tangents to the graph of $y = \sin x$ at (x_1, y_1) are in the following form

$$y = y_1 + m(x - x_1),$$

where m is the slope. The slope of the tangent at $(a, \sin a)$ is $\cos a$.

Considering the cosine function at the point $(a, \cos a)$ and using a similar approach, we get

$$\begin{aligned}\cos x &= \cos [a + (x - a)] \\ &= \cos a \cos (x - a) - \sin a \sin (x - a).\end{aligned}$$

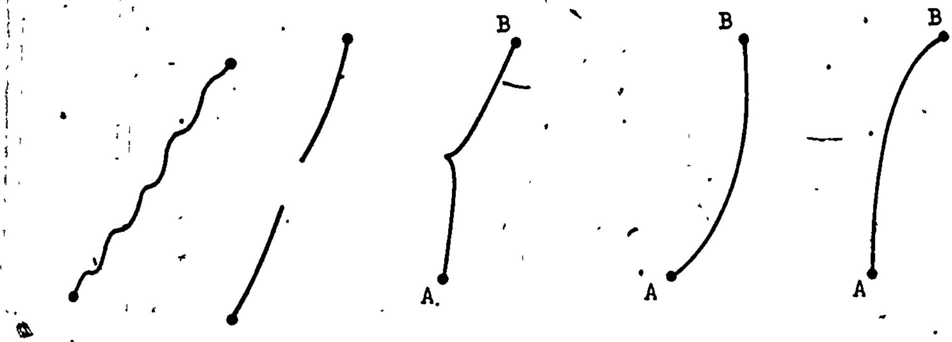
Replacing $\cos (x - a)$ and $\sin (x - a)$ by their best linear approximations 1 and $x - a$ respectively, we get for the equation of the tangent line at $(a, \cos a)$

$$y = \cos a - (\sin a)(x - a).$$

Thus, the slope of the tangent to the graph of the cosine function at $(a, \cos a)$ is $-\sin a$.

Throughout our discussion we obtain the graphs of functions by plotting points and connecting them with a smooth curve. This gives us a rough picture of the function but may leave unanswered for some Missouri students the question of whether this process is legitimate. How do we know that the graph between two points is really as smooth as we picture it to be? We could continue to plot points even closer together to get a more refined picture of the behavior of the function, but the tedious process would be less than satisfactory. Without interrupting the continuity of the text (and the thought process for most students) we discuss here the shape of the graph of $y = \sin x$, using our definition in terms of the unit circle and knowledge of the derivative to further justify the graphs we plotted in Section 3-3. But even this discussion is incomplete (e.g., we base our arguments on the intuitive geometrical ideas about arc length which we used to define the circular functions).

Here are five possibilities for the graph of $y = \sin x$ between the points $A(\frac{\pi}{6}, \sin\frac{\pi}{6})$ and $B(\frac{\pi}{4}, \sin\frac{\pi}{4})$.



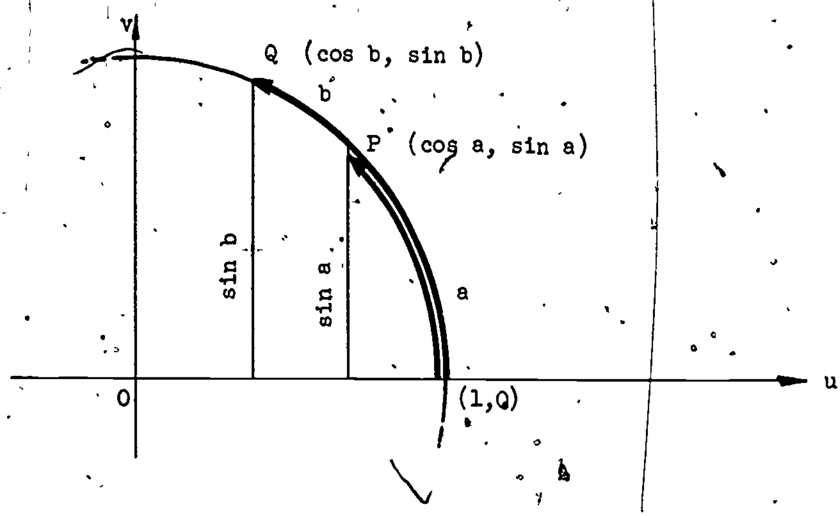
(i) oscillation | (ii) a gap | (iii) a corner | (iv) convex | (v) concave

Here we give simple tentative arguments to show that none of the first four pictures shown can be correct.

We can rule out the possibility of oscillation if we can show that the sine function is <u>increasing</u> between 0 and $\frac{\pi}{2}$. We need to show that

$$\text{if } 0 \leq a < b \leq \frac{\pi}{2}, \text{ then } \sin a < \sin b.$$

This statement is a simple consequence of the definition of $\sin x$ as the coordinate of a point which is x units around the unit circle from $(1,0)$. Since we are interested in the interval $\frac{\pi}{6} \leq x \leq \frac{\pi}{4}$, we can restrict our attention to the first quadrant of the unit circle. If a and b are between 0 and $\frac{\pi}{2}$ then $P(\cos a, \sin a)$ and $Q(\cos b, \sin b)$ must lie in the first quadrant. Furthermore, if $a < b$, then Q is above P, which means that $\sin a < \sin b$.



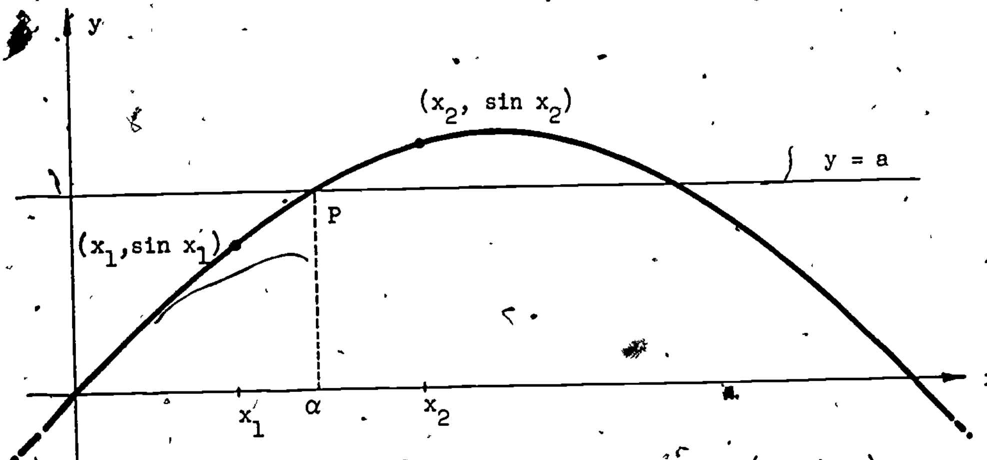
If $0 \le a < b \le \frac{\pi}{2}$ then Q is above P.

The fact that the sine function increases between 0 and $\frac{\pi}{2}$ eliminates the possibility that the graph of $y = \sin x$ can oscillate between $0 < \frac{\pi}{6} \le x \le \frac{\pi}{4} < \frac{\pi}{2}$. Thus we rule out Figure (i).

To show that Figure (ii) cannot be the graph of $y = \sin x$ over the interval $\frac{\pi}{6} \le x \le \frac{\pi}{4}$ we can show that the graph of the sine function has no "gaps." (A complete proof of this makes use of the fact that the real number line has no "gaps" and will be given in Appendix 7.) Here we use an intuitive geometric argument about arc length to rule out (ii). We can show that every horizontal line between $y = -1$ and $y = 1$ meets the graph of $y = \sin x$. If that is true then the graph has no "holes." More precisely we shall show that

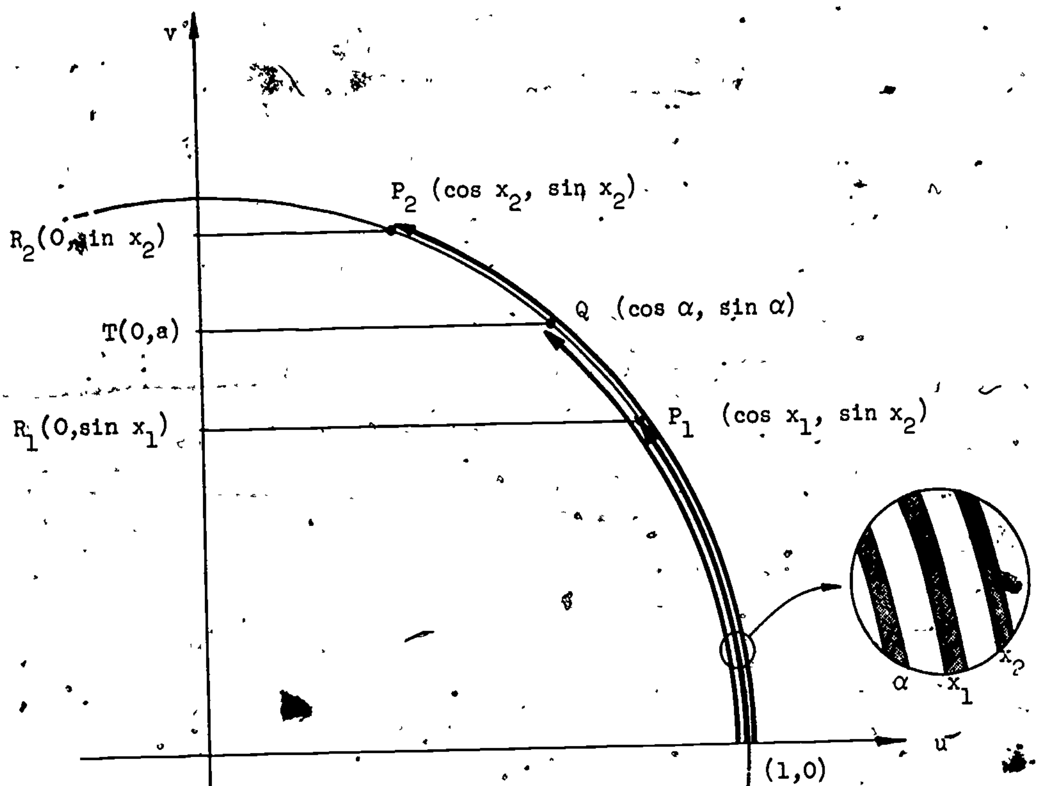
(3) if $x_1 < x_2$ and $\sin x_1 < a < \sin x_2$ then there is a number α, where $x_1 < \alpha < x_2$ such that $\sin \alpha = a$.

This amounts to showing that the graph cannot have a "hole" between $(x_1, \sin x_1)$ and $(x_2, \sin x_2)$.



There must be an α, $x_1 < \alpha < x_2$, such that $P(\alpha, \sin \alpha)$ lies on the graph of $y = \sin x$ and on the line $y = a$.

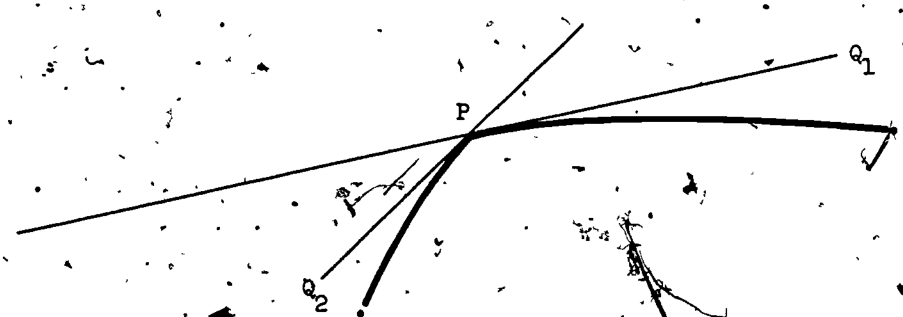
To establish (3), we again make use of the unit circle definition of the sine function. We draw arcs of lengths x_1 and x_2 units and the points $P_1(\cos x_1, \sin x_1)$ and $P_2(\cos x_2, \sin x_2)$. Horizontal lines through these points intersect the v-axis in the points $R_1(0, \sin x_1)$ and $R_2(0, \sin x_2)$.



The hypotheses of (3) tell us that $\sin x_1 < a < \sin x_2$ so that the point $T(0,a)$ lies between R_1 and R_2.

The horizontal line through T meets the unit circle in the point Q which lies between P_1 and P_2. The point Q is α arc units from $(1,0)$ so that Q has coordinates $(\cos\alpha, \sin\alpha)$. Now we see that $x_1 < \alpha < x_2$. Since the line TQ is horizontal, we must have $\sin\alpha = a$. Having established (3), we eliminate Figure (ii) as a possibility.

Now we show that the graph has no "corners"; i.e., that Figure (iii) cannot be a portion of the sine curve. If a graph has a corner point the set of slope approximations on the right and those to the left are not approximations to the same number; whence there can be no best linear approximation to a graph at a corner point.

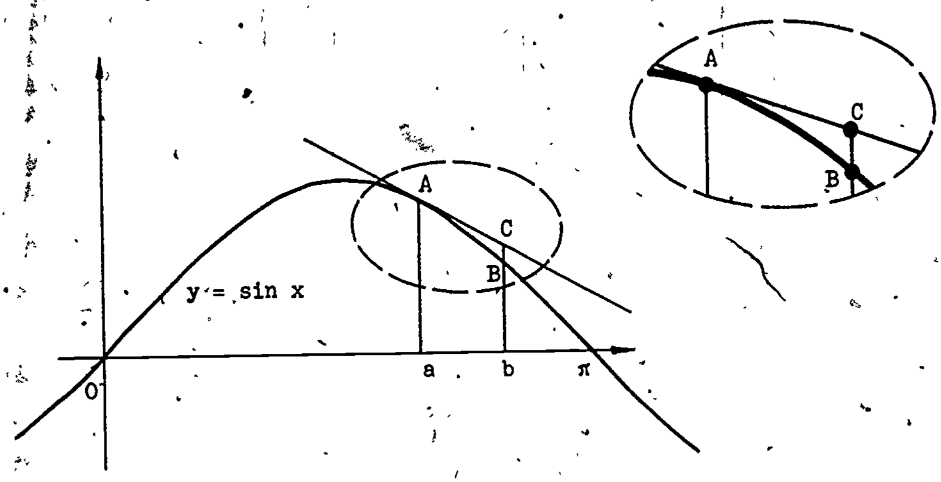


While a precise proof of this would lead us far astray, we note that the line PQ_1 approximates the curve to the right of P, while line PQ_2 approximates the curve to the left of P; but there is no line which "best fits" the curve at P on both sides of P. Since we know that at each point of the sine curve there is a line of "best fit," the sine curve can have no "corners."

To complete our discussion we must eliminate Figure (iv) and show that (v) is the best picture. We can show that between 0 and π, the graph of $y = \sin x$ must be concave; that is, the curve lies below the tangent line at any point in this interval. Suppose that

(4) $$0 \leq a < b \leq \pi.$$

We need to show that the point C lies above point B.



From (2) we know that the tangent line at A, has the equation

$$y = \sin a + (\cos a)(x - a).$$

Since C has abscissa b and lies on this tangent line, the ordinate of C must be

$$\sin a + (\cos a)(b - a).$$

Since B is a point of the graph of $y = \sin x$ the second ordinate of B is $\sin b$. To show that B lies below C we need to show that

$$\sin b < \sin a + (\cos a)(b - a).$$

This amounts to establishing that

(5) $$\sin b - \sin a < (b - a) \cos a.$$

To prove inequality (5) we write b and a as

$$b = \frac{(b + a) + (b - a)}{2} \quad \text{and} \quad a = \frac{(b + a) - (b - a)}{2}.$$

Now we use addition formula (6) of Section 3-5 to obtain

$$\sin b = \sin \frac{(b + a) + (b - a)}{2} = \sin \left(\frac{b+a}{2}\right) \cos \left(\frac{b-a}{2}\right) + \cos \left(\frac{b+a}{2}\right) \sin \left(\frac{b-a}{2}\right)$$

and

$$\sin a = \sin \frac{(b + a) - (b - a)}{2} = \sin \left(\frac{b+a}{2}\right) \cos \left(\frac{b-a}{2}\right) - \cos \left(\frac{b+a}{2}\right) \sin \left(\frac{b-a}{2}\right).$$

Subtracting $\sin a$ from $\sin b$, we get

(6) $$\sin b - \sin a = 2 \cos \left(\frac{b + a}{2}\right) \sin \left(\frac{b - a}{2}\right).$$

Since we have assumed (4) that $b > a$, and we know (from (8) of Section 4-1)

that $\sin x < x$ for $x > 0$, we can write

(7) $$2 \sin \left(\frac{b-a}{2}\right) < b - a.$$

Again using (4) the fact that $a < b$, we have

$$a < \frac{b+a}{2} < b.$$

In a manner similar to that used to establish that $\sin a < \sin b$, we could show that the cosine function decreases on the interval $0 \leq x \leq \pi$. Using this fact, we can write

(8) $$\cos \frac{b+a}{2} < \cos a.$$

From (6) we have

$$\sin b - \sin a = 2 \sin \left(\frac{b-a}{2}\right) \cos \left(\frac{b+a}{2}\right).$$

Now using (7) and (8) we can write

$$\sin b - \sin a < (b - a) \cos a.$$

Thus, we have shown that if $0 \leq a < b \leq \pi$, then

$$\sin b - \sin a < (b - a) \cos a.$$

A similar argument establishes the same inequality for $0 \leq b < a \leq \pi$ and completes the proof that the graph of the sine function is concave on the interval $0 \leq x \leq \pi$.

A student may wonder (in Section 4-2) why we appeal to a geometric discussion to evaluate $\lim_{h \to 0} \frac{\sin h}{h}$. In Chapter 3 we employed a diagram to define circular functions and to obtain the basic relationships of trigonometry. The student might expect that we are now in a position to evaluate this limit by formal computation from the relationships of Chapter 3 with no further appeal to intuitive geometric discussion. The reason this cannot be done is that the trigonometric identities of Chapter 3 have the same form, no matter what unit is used for angle measure. The crucial inequality

$$\cos x < \frac{\sin x}{x} < 1$$

is the first place where angle measure appears both as an argument of a trigonometric function and independently. Thus the inequality depends on the unit of measure; it holds specifically for radian measure and does not hold for any other measure such as degree measure.

Recall that there are three closely related, though distinct, kinds of trigonometric functions passing under the same name. First, there are the trigonometric functions of geometrical objects, namely, the angles introduced in geometry. Then, when we introduce an angle measure, the functions are functions of a real variable. The real functions depend upon the measure of the angles. Thus, the numerical functions obtained by measuring angles in degrees (a relic of the Babylonian sexagesimal numeration) are not the same as the functions obtained by measuring the angle in radians. Since an angle measured by x degrees is measured by $\frac{\pi}{180} x$ radians, a trigonometric function, say the sine function, defined in terms of degree measure is related to the corresponding trigonometric function defined in terms of radian measure by

$$\sin^{\circ} x = \sin \frac{\pi}{180} x.$$

The choice to include or to omit the foregoing Teacher's Commentary material in the Student's Text reflects a dilemma well known to teachers and writers.

On nearly every page of this text, we have found it necessary to choose between logical mathematical development and what we believe to be pedagogically sound exposition. Seldom is it possible to catch both rabbits, but we chase them. We try to be honest; e.g., to distinguish between a proof (in a formal mathematical sense), and an heuristic geometrical argument or plausibility test.

We attempt to nurture a student's intuition while cultivating an inquiring attitude. If we err on the side of handwaving at this stage in the text, we try in the appendices and Teacher's Commentary to indicate how logical gaps can be filled, should penetrating questions be raised by a Missouri student. As for students from other states, we try not to distract them: we avoid raising issues that would otherwise never occur to them at their present level of maturity. (We believe, for example, that to worry too early about the continuity of functions is to obstruct the continuity of the thought process.) We will raise a question that might not occur to the student if that question seems germane to the central ideas under consideration. But in general we adhere to the adage that before one begins to split hairs he must grow some.

Furthermore, we try to resist the temptation to tell a student all we know about a topic (no matter how short a time that might take). We sympathize with the student who began a book report with the following statements: "I have just read a book about penguins. The book tells more about penguins than anyone would ever care to know."

Solutions Exercises 4-1

1. (a) equation: $g = 1$

   (b) slope: $m = 0$

   (c) By definition of derivative, the limit is the slope of the tangent to the graph of $y = \cos x$ at $(0,1)$; $m = 0$.

2. (a) equation: $y = x$

   (b) slope: $m = 1$

   (c) By definition of derivative, the limit is the slope of the tangent to the graph of $y = \sin x$ at $(0,0)$; $m = 1$.

3. (a) $\lim\limits_{h \to 0} \dfrac{\cos h - 1}{h} = 0$

   (b) $\lim\limits_{h \to 0} \dfrac{\sin h}{h} = 1$

4. (a) $m(PQ) = \dfrac{\cos x - 1}{x}$

   (b)

B (0,1)	Q_i $(x_i, \cos x_i)$	$m(PQ_i)$ $\dfrac{\cos x_i - 1}{x_i}$	Eqn. line PQ_i $y = 1 - mx$
Q_1 :	(.5, .87758)	−.245	$y = 1 - .245x$
Q_2 :	(.4, .92106)	−.197	$y = 1 - .197x$
Q_3 :	(.3, .95534)	−.149	$y = 1 - .149x$
Q_4 :	(.2, .98007)	−.0997	$y = 1 - .0997x$
Q_5 :	(.1, .99500)	−.05	$y = 1 - .05x$
Q_6 :	(.01, .99995)	−.005	$y = 1 - .005x$

5. (a) $m(PQ) = \frac{\sin x}{x}$

(b)

P (0,0)	Q_i $(x_i, \sin x_i)$	$m(PQ_i)$ $\frac{\sin x}{x}$	Eqn. line PQ_i $y = mx$
Q_1 :	(.5, .47943)	.9589	$y \doteq .9589\,x$
Q_2 :	(.4, .38942)	.9736	$y = .9736\,x$
Q_3 :	(.3, .29552)	.9851	$y = .98507\,x$
Q_4 :	(.2, .19867)	.9934	$y = .9934\,x$
Q_5 :	(.1, .09983)	.9983	$y = .9983\,x$
Q_6 :	(.01, .01000)	1.0000^-	$y = 1.0000^-\,x$

6. (a) $x \cos x < \sin x$ (10)

$\sin x < x$ (8)

$\therefore x \cos x < \sin x < x$

(b) (i) $1 - \cos x \leq \frac{x^2}{2}$ (4)

$x > 0, \therefore \frac{1 - \cos x}{x} < \frac{x}{2}$

(ii) $1 - \cos x \leq \frac{x^2}{2}$ (4)

$x < 0, \therefore \frac{1 - \cos x}{x} > \frac{\pi}{2}$

(iii) Combining (i) and (ii)

$x \neq 0, \left|\frac{1 - \cos x}{x}\right| < \frac{|x|}{2}$

(c) $1 - \cos x \leq \frac{x^2}{2}$ (4)

$\therefore \cos x \geq 1 - \frac{x^2}{2}$

$x > 0, x \cos x \geq x - \frac{x^3}{2}$

But $\sin x > x \cos x$ (10)

$\therefore \sin x \geq x - \frac{x^3}{2}$; $0 < x < \frac{\pi}{2}$

(d) $x \cos x < \sin x < x$

$x > 0$, $\cos x < \frac{\sin x}{x} < 1$

$\therefore -\cos x > -\frac{\sin x}{x} > -1$

$1 - \cos x > 1 - \frac{\sin x}{x} > 0$

$x < 0$, $\cos(-x) = \cos x$; $\frac{\sin(-x)}{(-x)} = \frac{\sin x}{x}$

$\therefore 0 < |x| < \frac{\pi}{2}$, $1 - \frac{\sin x}{x} < 1 - \cos x$

7. We are given that

(1) $0 \le 1 - \cos h \le \frac{h^2}{2}$,

(2) $h \cos h < \sin h < h$, for $0 < h < \frac{\pi}{2}$,

From (1) we get

$$1 - \frac{h^2}{2} < \cos h \le 1.$$

We substitute this into (2) to obtain

$$h\left(1 - \frac{h^2}{2}\right) < \sin h < h, \quad \text{for} \quad 0 < h < \frac{\pi}{2}.$$

Dividing by h, we get

$$1 - \frac{h^2}{2} < \frac{\sin h}{h} < 1, \quad \text{for} \quad 0 < h < \frac{\pi}{2}.$$

If we replace h by $-h$ our terms are unaffected since

$$(-h)^2 = h^2 \quad \text{and} \quad \frac{\sin(-h)}{(-h)} = \frac{-\sin h}{-h} = \frac{\sin h}{h}.$$

Consequently, we can write

$$1 - \frac{h^2}{2} < \frac{\sin h}{h} < 1, \quad \text{for} \quad 0 < |h| < \frac{\pi}{2}.$$

8. A similar argument to the solution of Number 7, using (1) of Number 7, $0 \le 1 - \cos h \le \frac{h^2}{2}$, we have, for $h > 0$, $0 < \frac{1 - \cos h}{h} < \frac{h}{2}$; and, for $h < 0$, $0 > \frac{1 - \cos h}{h} > \frac{h}{2}$. Therefore,

$$\left|\frac{1 - \cos h}{h}\right| < \frac{|h|}{2}, \quad \text{for} \quad h \ne 0.$$

9. If $h = 0.01$ then $\frac{h^2}{2} = 0.00005$. From Number 7 we have

$$0.99995 < \frac{\sin(0.01)}{0.01} < 1.$$

From Number 8, we get

$$-0.005 \leq \frac{1 - \cos(0.01)}{0.01} \leq 0.005.$$

Letting $h = -0.001$, we have $\frac{h^2}{2} = 0.0000005$, so that from Number 7 we get

$$0.9999995 < \frac{\sin(-0.001)}{-0.001} < 1;$$

and from Number 8 we obtain

$$-0.0005 < \frac{1 - \cos(-0.001)}{-0.001} < 0.0005.$$

10. (a) The inequality (result of No. 7)

$$1 - \frac{h^2}{2} < \frac{\sin h}{h} < 1, \quad \text{for } 0 < |h| < \frac{\pi}{2},$$

tells us that the ratio $\frac{\sin h}{h}$ is squeezed between $1 - \frac{h^2}{2}$ and 1. If $|h|$ is small the quantity $1 - \frac{h^2}{2}$ is very close to 1. We can say that

$$\lim_{h \to 0} \frac{\sin h}{h} = 1.$$

(b) Similarly, the inequality (result of No. 8)

$$\left|\frac{1 - \cos h}{h}\right| < \frac{|h|}{2}, \quad \text{for } h \neq 0,$$

tells us that

$$\lim_{h \to 0} \frac{1 - \cos h}{h} = 0.$$

Solutions Exercises 4-2

1. (a) Using addition formula (4) of Section 3-5, we have

$$\frac{\cos (x + h) - \cos x}{h} = \frac{\cos x \cos h - \sin x \sin h - \cos x}{h}$$

$$= \cos x \left(\frac{\cos h - 1}{h}\right) - \sin x \left(\frac{\sin h}{h}\right)$$

(b) $$D(\cos x) = \lim_{h \to 0} \frac{\cos (x + h) - \cos x}{h}$$

$$= \lim_{h \to 0} \left[\cos x \frac{\cos h - 1}{h} - \sin x \frac{\sin h}{h}\right]$$

$$= \cos x\ (0) - \sin x\ (1)$$

$$= -\sin x.$$

2. (a) $$\sin (x + h) - \sin x = \sin (\alpha + \beta) - \sin (\alpha - \beta)$$

$$= 2 \cos \alpha \sin \beta$$

$\alpha + \beta = x + h$ $\qquad 2\alpha = 2x + h$ $\qquad 2\beta = h$

$\alpha - \beta = x$ $\qquad \alpha = x + \frac{h}{2}$ $\qquad \beta = \frac{h}{2}$

$$\sin (x + h) - \sin x = 2 \cos \left(x + \frac{h}{2}\right) \sin \left(\frac{h}{2}\right)$$

(b) $$D(\sin x) = \lim_{h \to 0} \frac{\sin (x + h) - \sin x}{h}$$

$$= \lim_{h \to 0} \frac{2}{h} \cos \left(x + \frac{h}{2}\right) \sin \left(\frac{h}{2}\right)$$

$$= \lim_{h \to 0} \cos \left(x + \frac{h}{2}\right) \frac{\sin \left(\frac{h}{2}\right)}{\left(\frac{h}{2}\right)}$$

$$= \cos x$$

3. (a) $$\cos (x + h) - \cos x = \cos (\alpha + \beta) - \cos (\alpha - \beta)$$

$$= -2 \sin \alpha \sin \beta$$

$$= -2 \sin \left(x + \frac{h}{2}\right) \sin \left(\frac{h}{2}\right)$$

(b) $D(\cos x) = \lim_{h \to 0} \frac{\cos (x + h) - \cos x}{h}$

$= \lim_{h \to 0} [- \frac{2}{h} \sin (x + \frac{h}{2}) \sin (\frac{h}{2})]$

$= - \left[\lim_{h \to 0} \sin (x + \frac{h}{2}) \frac{\sin (\frac{h}{2})}{(\frac{h}{2})}\right]$

$= -\sin x$

4. (a) $.9995 < \frac{\sin (.01)}{.01} < 1$

$.009995 \leq \sin (.01) < .01$

(b) $-.0005 < \frac{1 - \cos (-.001)}{(-.001)} < .0005$

$.0000005 > 1 - \cos (-.001) > -.0000005$

$-.9999995 > -\cos (-.001) > -1.0000005$

$.9999995 < \cos (-.001) < 1.0000005$

5. (a) $1 - \frac{h^2}{2} < \frac{\sin h}{h} < 1 \qquad 0 < |h| < \frac{\pi}{2}$

(i) Let $h = .1$:

$1 - \frac{.01}{2} < \frac{\sin .1}{(.1)} < 1$

$1 - .005 < \frac{\sin (.1)}{(.1)} < 1$; i.e., $\frac{\sin (.1)}{(.1)} \approx 1$

$.0995 < \sin (.1) < .1 \qquad \therefore \sin (.1) \approx .1$

(ii) Let $h = .01$:

$1 - \frac{.0001}{2} < \frac{\sin (.01)}{(.01)} < 1$

$1 - .00005 < \frac{\sin (.01)}{(.01)} < 1$; i.e., $\frac{\sin (.01)}{(.01)} \approx 1$

$.0099995 < \sin .01 < .01 \qquad \therefore \sin (.01) \approx .01$

(iii) Let $h = .001$:

$1 - \frac{.000001}{2} < \frac{\sin (.001)}{(.001)} < 1$

$1 - .0000005 < \frac{\sin (.001)}{(.001)} < 1$; i.e., $\frac{\sin (.001)}{(.001)} \approx 1$

$.0009999995 < \sin (.001) < .001 \quad \therefore \sin (.001) \approx .001$

(iv) Let $h = .0001$:

$$(.0001)\left(1 - \frac{.00000001}{2}\right) < \sin(.0001) < 1(.0001)$$

$$.0000999999995 < \sin(.0001) < .0001;$$

$$\text{i.e.,}\quad \frac{\sin(.0001)}{(.0001)} \approx 1$$

$$\therefore\ \sin(.0001) \approx .0001$$

(b) $1 - \frac{h^2}{2} \leq \cos h \leq 1$

(i) $h = .1$:

$$1 - \frac{.01}{2} \leq \cos .1 \leq 1$$

$$1 - .005 \leq \cos .1 \leq 1$$

$$.995 \leq \cos .1 \leq 1 \qquad \cos(.1) > .99$$

(ii) $h = .01$:

$$1 - \frac{.0001}{2} \leq \cos .01 \leq 1$$

$$1 - .00005 \leq \cos .01 \leq 1$$

$$.99995 \leq \cos .01 \leq 1 \qquad \cos(.01) > .9999$$

(iii) $h = .001$:

$$1 - \frac{.000001}{2} \leq \cos .001 \leq 1$$

$$1 - .0000005 \leq \cos .001 \leq 1$$

$$.9999995 \leq \cos .001 \leq 1 \qquad \cos(.001) > .999999$$

(iv) $h = .0001$:

$$.999999995 \leq \cos .0001 \leq 1 \qquad \cos(.0001) > .99999999$$

6. (a) $\left|\frac{1 - \cos h}{h}\right| \leq \frac{|h|}{2}$ $\qquad h \neq 0$

(i) $|h| = .1$ $\qquad \left|\frac{1 - \cos (.1)}{.1}\right| \leq \frac{|.1|}{2} = .05$

(ii) $|h| = .01$ $\qquad \left|\frac{1 - \cos (.01)}{.01}\right| \leq \frac{|.01|}{2} = .005$

(iii) $|h| = .001$ $\qquad \left|\frac{1 - \cos (.001)}{.001}\right| \leq \frac{|.001|}{2} = .0005$

(iv) $|h| = .0001$ $\qquad \left|\frac{1 - \cos (.0001)}{.0001}\right| \leq \frac{|.0001|}{2} = .00005$

(b) $1 - \frac{h^2}{2} < \frac{\sin h}{h} < 1$ $\qquad 0 < |h| < \frac{\pi}{2}$

Similar the solution of Number 5(a)

(i) Let $|h| = .1$: $\qquad .995 < \frac{\sin (.1)}{(.1)} < 1$

(ii) Let $|h| = .01$: $\qquad .99995 < \frac{\sin (.01)}{.01} < 1$

(iii) Let $|h| = .001$: $\qquad .9999995 < \frac{\sin (.001)}{.001} < 1$

(iv) Let $|h| = .0001$: $\qquad .999999995 < \frac{\sin (.0001)}{(.0001)} < 1$

7. (a) $\lim\limits_{x \to 0} \frac{\sin 3x}{2x}$

$$1 - \frac{h^2}{2} < \frac{\sin h}{h} < 1$$

$$1 - \frac{(3x)^2}{2} < \frac{\sin 3x}{3x} < 1$$

Multiplying by $\frac{3}{2}$, we get

$$\frac{3}{2} - \frac{3}{2}\frac{(3x)^2}{2} < \frac{\sin 3x}{2x} < \frac{3}{2}$$

$$\frac{3}{2} - \frac{27}{4}x^2 < \frac{\sin 3x}{2x} < \frac{3}{2}$$

$\frac{\sin 3x}{2x}$ is "squeezed" between $\frac{3}{2} - \frac{27}{4}x^2$ and $\frac{3}{2}$.

Since $\frac{3}{2} - \frac{27}{4}x^2$ is close to $\frac{3}{2}$ when x is small, we can conclude that the limit of $\frac{\sin 3x}{2x}$ as x goes to zero is $\frac{3}{2}$.

(b) Either $\lim_{x \to 0} \left(\frac{\sin 3x}{2x}\right)$ Let $3x = h$

$$\lim_{\frac{h}{3} \to 0} \left(\frac{\sin h}{\frac{2}{3} h}\right) \qquad \therefore \quad x = \frac{h}{3}, \quad 2x = \frac{2}{3}h$$

$$\lim_{\frac{h}{3} \to 0} \left(\frac{3}{2}\right)\left(\frac{\sin h}{h}\right) = \frac{3}{2} \cdot 1 = \frac{3}{2}$$

As $x \to 0$, $\frac{h}{3} \to 0$ and $h \to 0$.

or without substitution

$$\lim_{x \to 0} \frac{\sin 3x}{2x}$$

$$\lim_{x \to 0} \frac{\sin 3x}{\left(\frac{2}{3}\right)3x} = \lim_{x \to 0} \left(\frac{3}{2}\right)\left(\frac{\sin 3x}{3x}\right) = \frac{3}{2} \cdot 1 = \frac{3}{2}$$

8. (a) $\lim_{x \to 0} \frac{\sin x}{3x} = \lim_{x \to 0} \left(\frac{1}{3}\right)\left(\frac{\sin x}{x}\right) = \frac{1}{3} \cdot 1 = \frac{1}{3}$

(b) $\lim_{x \to 0} \frac{\sin^2 x}{x} = \lim_{x \to 0} (\sin x)\left(\frac{\sin x}{x}\right) = 0 \cdot 1 = 0$

(c) $\lim_{x \to 0} \frac{\tan x}{x} = \lim_{x \to 0} \left(\frac{\sin x}{x}\right)\left(\frac{1}{\cos x}\right) = 1 \cdot 1 = 1$

(d) $\lim_{h \to 0} \frac{h}{\sin h} = \lim_{h \to 0} \frac{1}{\frac{\sin h}{h}}$

$$= \frac{1}{1} = 1$$

(e) $\lim_{x \to 0} \frac{\sin 7x}{\sin 3x} = \lim_{x \to 0} \frac{7}{3}\left(\frac{\sin 7x}{7x}\right)\left(\frac{3x}{\sin 3x}\right) = \frac{7}{3} \cdot 1 \cdot 1 = \frac{7}{3}$

(f) $\lim_{x \to 0} \frac{\cos 7x}{\cos 3x} = 1.$

(g) $\lim_{\theta \to 0} \frac{1 - \cos \theta}{\theta^2} \cdot \frac{1 + \cos \theta}{1 + \cos \theta} = \lim_{\theta \to 0} \frac{\sin^2 \theta}{\theta^2(1 + \cos \theta)}$

$$= \lim_{\theta \to 0} \left(\frac{\sin \theta}{\theta}\right)^2\left(\frac{1}{1 + \cos \theta}\right)$$

$$= 1 \cdot \frac{1}{2} = \frac{1}{2}$$

(h) $\lim_{\theta \to 0} \left(\frac{\tan 2x}{\sin x}\right) = \lim_{\theta \to 0} \left(\frac{\sin 2x}{\cos 2x} \cdot \frac{1}{\sin x}\right)$

$= \lim_{\theta \to 0} \left(\frac{\sin 2x}{2x}\right)\left(\frac{x}{\sin x}\right)\left(\frac{1}{\cos 2x}\right)(2)$

$= 1 \cdot 1 \cdot 1 \cdot 2 = 2$

(i) $\lim_{t \to 0} \left(\frac{\sin 2t}{2t^2 + t}\right) = \lim_{t \to 0} \left(\frac{\sin 2t}{2t(t + \frac{1}{2})}\right)$

$= \lim_{t \to 0} \left(\frac{\sin 2t}{2t}\right)\left(\frac{1}{t + \frac{1}{2}}\right) = 1 \cdot 2 = 2$

(j) $\lim_{\theta \to 0} \theta \cot 2\theta = \lim_{\theta \to 0} \left(\frac{2\theta}{\sin 2\theta}\right) \cdot \left(\frac{\cos 2\theta}{2}\right) = 1 \cdot \frac{1}{2} = \frac{1}{2}$

(k) $\lim_{\theta \to \frac{\pi}{2}} \left(\frac{\cos \theta}{\frac{\pi}{2} - \theta}\right)$ Let $\cos \theta = \sin\left(\frac{\pi}{2} - \theta\right)$.

$\lim_{\left(\theta - \frac{\pi}{2}\right) \to 0} \left(\frac{\sin\left(\frac{\pi}{2} - \theta\right)}{\frac{\pi}{2} - \theta}\right)$

$\lim_{\left(\theta - \frac{\pi}{2}\right) \to 0} \left(\frac{\sin\left(\theta - \frac{\pi}{2}\right)}{\left(\theta - \frac{\pi}{2}\right)}\right)$ $\frac{\sin(-x)}{(-x)} = \frac{\sin x}{x}$

$\lim_{\alpha \to 0} \left(\frac{\sin \alpha}{\alpha}\right) = 1$ Let $\alpha = \theta - \frac{\pi}{2}$

(ℓ) $\lim_{x \to 0} \frac{\sin 5x - \sin 3x}{x}$

$\lim_{x \to 0} \left(\frac{\sin 5x}{x} - \frac{\sin 3x}{x}\right)$

$\lim_{x \to 0} \left[5\left(\frac{\sin 5x}{5x}\right) - 3\left(\frac{\sin 3x}{3x}\right)\right] = 5 \cdot 1 - 3 \cdot 1 = 2$

9. (a) If $f : x \to \sin x$, then, by definition,

$$\lim_{x \to a} \frac{\sin x - \sin a}{x - a} = f'(a) = \cos a.$$

(b) Using part (a) we have

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{\sin x - \sin 0}{x - 0}$$

$$= \cos 0$$

$$= 1.$$

10. $f(x) = \sin x$ $\qquad f'(x) = \cos x$

(a) (i) Slope of $\sin x$ at $x = \frac{\pi}{3}$: $f'(\frac{\pi}{3}) = \frac{1}{2}$

(ii) Slope of $\sin x$ at $x = \frac{\pi}{6}$: $f'(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$

(iii) Slope of $\sin x$ at $x = \frac{3\pi}{4}$: $f'(\frac{3\pi}{4}) = -\frac{1}{\sqrt{2}}$

(iv) Slope of $\sin x$ at $x = \pi$: $f'(\pi) = -1$

(v) Slope of $\sin x$ at $x = \frac{3\pi}{2}$: $f'(\frac{3\pi}{2}) = 0$

(vi) Slope of $\sin x$ at $x = 0$: $f'(0) = 1$.

(b) Point $P(x_1, y_1)$

	x_1	$y_1 = f(x_1)$	$f'(x_1)$	Eqn. of line tangent to $y = \sin x$ at $x = x_1$
(i)	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$y = \frac{\sqrt{3}}{2} + \frac{1}{2}(x - \frac{\pi}{3})$ or $y = \frac{1}{2}x + \frac{3\sqrt{3} - \pi}{6}$
(ii)	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$y = \frac{1}{2} + \frac{\sqrt{3}}{2}(x - \frac{\pi}{6})$, or $y = \frac{\sqrt{3}}{2}x + \frac{6 - \pi\sqrt{3}}{12}$
(iii)	$\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	$y = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}(x - \frac{3\pi}{2})$ or $y = -\frac{1}{\sqrt{2}}x + \frac{2 + 3\pi}{2\sqrt{2}}$
(iv)	π	0	-1	$y = 0 - 1(x - \pi)$ or $y = -x + \pi$
(v)	$\frac{3\pi}{2}$	-1	0	$y = -1 + 0(x - \frac{3\pi}{2})$ or $y = -1$
(vi)	0	0	1	$y = 0 + 1(x - 0)$ or $y = x$

11. $f(x) = \cos x$ $\qquad$ $f'(x) = -\sin x$

(a) (i) Slope of $\cos x$ at $x = \frac{\pi}{6}$: $f'(\frac{\pi}{6}) = -\frac{1}{2}$

(ii) Slope of $\cos x$ at $x = \frac{2\pi}{3}$: $f'(\frac{2\pi}{3}) = -\frac{\sqrt{3}}{2}$

(iii) Slope of $\cos x$ at $x = -\frac{\pi}{4}$: $f'(-\frac{\pi}{4}) = \frac{1}{\sqrt{2}}$

(iv) Slope of $\cos x$ at $x = 0$: $f'(0) = 0$

(b)

	x_1	$f(x_1)$	$f'(x_1)$	Eqn. of tangent line
(i)	$\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$y = \frac{\sqrt{3}}{2} - \frac{1}{2}(x - \frac{\pi}{6})$ or $y = -\frac{1}{2}x + \frac{6\sqrt{3} + \pi}{12}$
(ii)	$\frac{2\pi}{3}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$y = -\frac{1}{2} - \frac{\sqrt{3}}{2}(x - \frac{2\pi}{3})$ or $y = -\frac{\sqrt{3}}{2}x + \frac{-3 + 2\pi\sqrt{3}}{6}$
(iii)	$-\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	$y = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}(x + \frac{\pi}{4})$ or $y = \frac{1}{\sqrt{2}}x + \frac{4 + \pi}{4\sqrt{2}}$
(iv)	0	1	0	$y = 1 + 0(x - 0)$ or $y = 1$

12. (a) $f = \sin$; $f' = \cos$

If tangent is horizontal, $f' = 0$.

$\therefore$ $\cos x = 0$, and $x = \frac{\pi}{2} + n\pi$, n integer.

(b) $f = \cos$; $f' = -\sin$

If tangent is horizontal, $f' = 0$.

$\therefore$ $-\sin x = 0$, and $x = n\pi$, n integer.

13. (a) $f = \sin$; $f' = \cos$

If tangent is parallel to line $y = x$, $f' = 1$

$\therefore$ $\cos x = 1$, and $x = 2n\pi$, n integer.

(b) $f = \cos$; $f' = -\sin$

If tangent is parallel to line $y = x$, $f' = 1$

$\therefore$ $-\sin x = 1$, $\sin x = -1$, and $x = \frac{3\pi}{2} + 2n\pi$, n integer.

14. (a) $f = \sin$; $f' = \cos$

$\therefore$ $\cos x = -1$, and $x = \pi + 2n\pi$, n integer.

(b) $f = \cos$; $f' = -\sin$

$\therefore$ $-\sin x = -1$, $\sin x = 1$, and $x = \frac{\pi}{2} + 2n\pi$, n integer.

15. (a) $f = \sin$; $f' = \cos$

$\therefore \cos x = \frac{1}{2}$, and $x = \frac{\pi}{3} + 2\pi n, \ \frac{5\pi}{3} + 2\pi n$, n integer.

(b) $f = \cos$; $f' = -\sin$

$\therefore -\sin x = \frac{1}{2}$, $\sin x = -\frac{1}{2}$, and $x = \frac{7\pi}{6} + 2n\pi, \ \frac{11\pi}{6} + 2n\pi$, n integer.

16. (a) $f'(600\pi) = \cos 600\pi = \cos 0 = 1$

$f'(-200\pi - \frac{\pi}{6}) = \cos(-\frac{\pi}{6}) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$

$f'(60\pi - \frac{5\pi}{4}) = \cos(-\frac{5\pi}{4}) = \cos\frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$

(b) $g'(600\pi) = -\sin(600\pi) = -\sin 0 = 0$

$g'(-200\pi - \frac{\pi}{6}) = -\sin(-\frac{\pi}{6}) = \sin\frac{\pi}{6} = \frac{1}{2}$

$g'(60\pi - \frac{5\pi}{4}) = -\sin(-\frac{5\pi}{4}) = \sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$

17. (a) $f : x \to \sin x$

$f' : x \to \cos x$, i.e., $h : x \to \cos x$

$h' : x \to -\sin x$

$\therefore h' = -f$

(b) $f : x \to \sin x$ $\qquad$ $g : x \to \cos x$

$\begin{cases} f' : x \to \cos x \\ h : x \to \cos x \end{cases}$ $\qquad$ $\begin{cases} g' : x \to -\sin x \\ j : x \to -\sin x \end{cases}$

$h' : x \to -\sin x$ $\qquad$ $j' : x \to -\cos x$

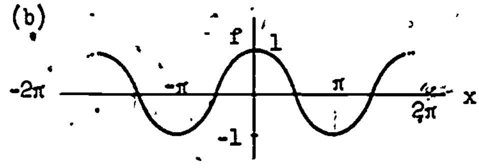
$\therefore h' = j = -\sin x$ and $h = -j' = \cos x$.

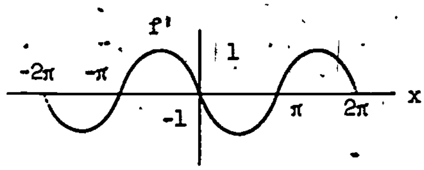
18. $f : x \to \cos x$ $\qquad$ $f' : x \to -\sin x$

(a) The function is increasing when $f' > 0$; i.e., f is increasing for $-\pi < x < 0$, $\pi < x < 2\pi$.

The function is decreasing when $f' \leq 0$; i.e., f is decreasing for $-2\pi \leq x < -\pi$, $0 < x < \pi$.

(b)





19. (a) The result of 1(a) tells us that $\underline{\cos u}$ is increasing and decreasing in the following intervals for the indicated values of $\underline{u}$:

$$\begin{cases} \text{dec.: } & -2\pi < u < -\pi \\ \text{incr.: } & -\pi < u < 0 \\ \text{dec.: } & 0 < u < \pi \\ \text{incr.: } & \pi < u < 2\pi \end{cases}$$

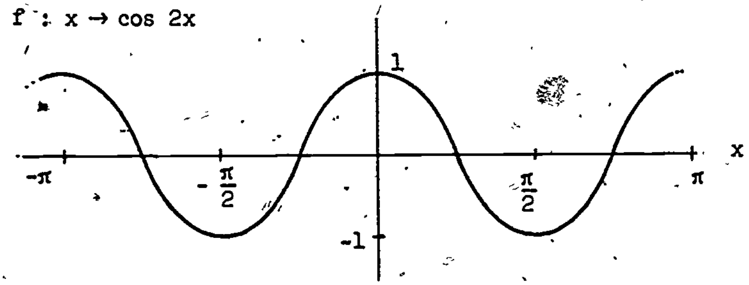
This means that $\underline{\cos 2x}$ is increasing and decreasing in the same intervals for the indicated values of $\underline{2x}$:

$$\begin{cases} \text{dec.: } & -2\pi < 2x < -\pi \\ \text{incr.: } & -\pi < 2x < 0 \\ \text{dec.: } & 0 < 2x < \pi \\ \text{incr.: } & \pi < 2x < 2\pi \end{cases}$$

Then it follows that $\underline{\cos 2x}$ is increasing and decreasing for the indicated values of $\underline{x}$:

$$\begin{cases} \text{dec.: } & -\pi < x < -\frac{\pi}{2} \\ \text{incr.: } & -\frac{\pi}{2} < x < 0 \\ \text{dec.: } & 0 < x < \frac{\pi}{2} \\ \text{incr.: } & \frac{\pi}{2} < x < \pi \end{cases}$$

(b) $f : x \to \cos 2x$



20. (a) $\cos\left(2x + \frac{\pi}{2}\right)$ Let $u = 2x + \frac{\pi}{2}$.

From the solution to Exercise 1(a) we know that $\underline{\cos u}$ is increasing and is decreasing for the indicated value of $\underline{u}$:

$$\begin{cases} \text{dec.: } & -2\pi < u < -\pi \\ \text{incr.: } & -\pi < u < 0 \\ \text{dec.: } & 0 < u < \pi \\ \text{incr.: } & \pi < u < 2\pi \end{cases}$$

i.e., $\underline{\cos\left(2x + \frac{\pi}{2}\right)}$ is increasing and is decreasing for the indicated values of $\underline{2x + \frac{\pi}{2}}$:

$$\begin{cases} \text{dec.: } & -2\pi < 2x + \frac{\pi}{2} < -\pi \\ \text{incr.: } & -\pi < 2x + \frac{\pi}{2} < 0 \\ \text{dec.: } & 0 < 2x + \frac{\pi}{2} < \pi \\ \text{incr.: } & \pi < 2x + \frac{\pi}{2} < 2\pi \end{cases}$$

$\cos\left(2x + \frac{\pi}{2}\right)$ is increasing and is decreasing for the indicated values of $x + \frac{\pi}{4}$:

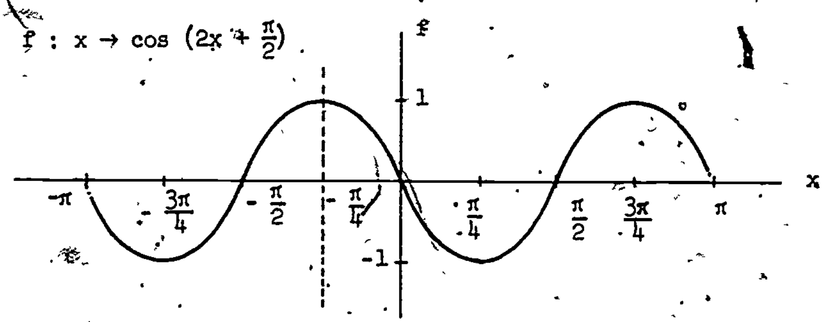
$$\left\{\begin{array}{ll} \text{dec.:} & -\pi < x + \frac{\pi}{4} < -\frac{\pi}{2} \\ \text{incr.:} & -\frac{\pi}{2} < x + \frac{\pi}{4} < 0 \\ \text{dec.:} & 0 < x + \frac{\pi}{4} < \frac{\pi}{2} \\ \text{incr.:} & \frac{\pi}{2} < x + \frac{\pi}{4} < \pi, \end{array}\right.$$

and $\cos\left(2x + \frac{\pi}{2}\right)$ is increasing and is decreasing for the indicated values of x.

$$\left\{\begin{array}{ll} \text{dec.:} & -\frac{5\pi}{4} < x < -\frac{3\pi}{4} \\ \text{incr.:} & -\frac{3\pi}{4} < x < -\frac{\pi}{4} \\ \text{dec.:} & -\frac{\pi}{4} < x < \frac{\pi}{4} \\ \text{incr.:} & \frac{\pi}{4} < x < \frac{3\pi}{4} \\ \text{dec.:} & \frac{3\pi}{4} < x < \frac{5\pi}{4} \end{array}\right.$$

Therefore, for the interval $|x| < \pi$, $\cos\left(2x + \frac{\pi}{2}\right)$ is decreasing for $|x| < \frac{\pi}{4}$ and for $\frac{3\pi}{4} < |x| < \pi$, and is increasing for $\frac{\pi}{4} < |x| < \frac{3\pi}{4}$.

(b) $f : x \to \cos\left(2x + \frac{\pi}{2}\right)$



21.

	Intervals			
	$(0, \frac{\pi}{2})$	$(\frac{\pi}{2}, \pi)$	$(\pi, \frac{3\pi}{2})$	$(\frac{3\pi}{2}, 2\pi)$
sin x	increasing	decreasing	decreasing	increasing
cos x	decreasing	decreasing	increasing	increasing

decreasing: $\frac{\pi}{2} < x < \pi$

increasing: $\frac{3\pi}{2} < x < 2\pi$

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22.

	Intervals $(0, \frac{\pi}{4})$	$(\frac{\pi}{4}, \frac{\pi}{2})$	$(\frac{\pi}{2}, \frac{3\pi}{4})$	$(\frac{3\pi}{4}, \pi)$	$(\pi, \frac{5\pi}{4})$	$(\frac{5\pi}{4}, \frac{3\pi}{2})$	$(\frac{3\pi}{2}, \frac{7\pi}{4})$	$(\frac{7\pi}{4}, 2\pi)$
(a) $\sin \frac{x}{2}$	increasing				decreasing			
(b) $\sin x$	increasing		decreasing				increasing	
(c) $\sin 2x$	incr.	decreasing		increasing		decreasing		incr.

decreasing

$\frac{5\pi}{4} < x < \frac{3\pi}{2}$;

increasing

$0 < x < \frac{\pi}{4}$

23\. (a) $f(x) = ax - \sin x$

$f'(x) = a - \cos x > 0$.

i.e., $\cos x < a$ $\quad \therefore \; a > 1$

(b) $f(x) = ax + \cos x$

$f'(x) = a - \sin x > 0$

i.e., $\sin x < a$, $\quad \therefore \; a > 1$

24\. (a) Coordinates:

$P(.4, \sin .4) = (.4, .38942)$

$D(.5, \sin .5) = (.5, .47943)$

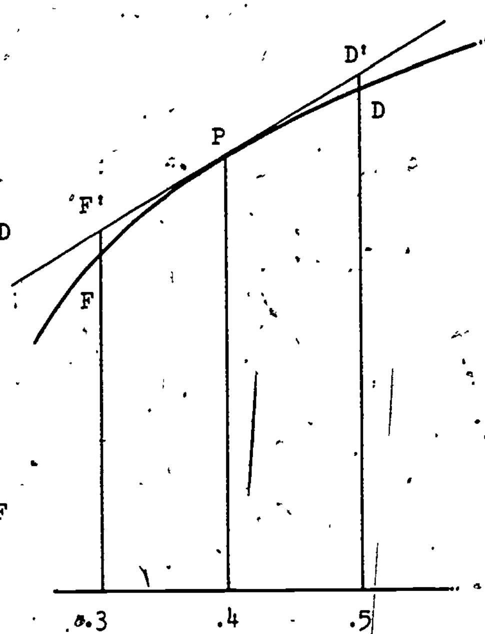
$D'(.5, \sin .4 + (.1) \cos .4)$

$$\left\{\begin{array}{l} \text{Ordinate of } D' \overset{?}{>} \text{ ordinate of } D \\ \sin .4 + (.1) \cos .4 \overset{?}{>} .47943 \\ .38942 + .092106 \overset{?}{>} .47943 \\ \qquad .48153 > .47943 \end{array}\right.$$

$F(.3, \sin .3) = (.3, .29552)$

$F'[.3, \sin .4 + (-.1) \cos .4]$,

$$\left\{\begin{array}{l} \text{Ordinate of } F' \overset{?}{>} \text{ ordinate of } F \\ \sin .4 + (-.1) \cos .4 \overset{?}{>} .29552 \\ .38942 - .092106 \overset{?}{>} .29552 \\ \qquad .29731 > .29552 \end{array}\right.$$



At $x = .3$ and $x = .5$

$\therefore$ The curve lies below the tangent to the curve at $P(.4, \sin .4)$

Tan. line:

$y = \sin .4 + \cos .4(x - .4)$

$y = .38942 - (.92106)(x - .4)$

(b) $Q[-.4, \sin(-.4)]$

$(-.4, -\sin .4)$

$(-.4, -.38942)$

$R[-.3, \sin(-.3)]$

$(-.3, -\sin .3)$

$(-.3, -.29552)$

$R'[-.3, -\sin .4 + (.1)\cos .4]$

abs. value of ordinate of R' > abs. value of ordinate of R

$|-\sin .4 + (.1)\cos .4| > |-.29552|$

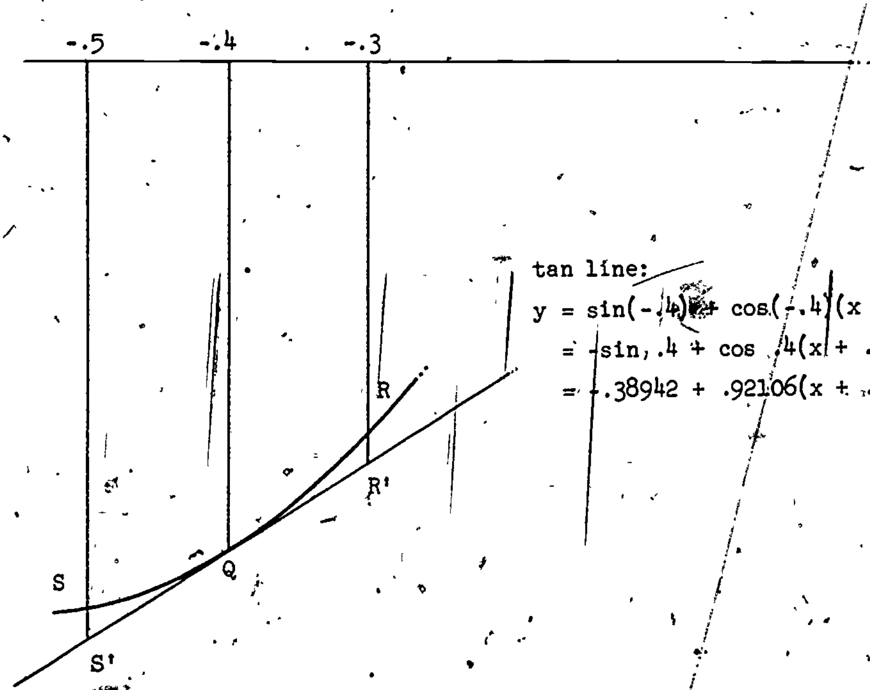
$|-.92106 + .092106| > |-.29552|$

abs. value of ord. of S' > abs. value of ordinate of S

$|1 - \sin .4 + (-.1)\cos .4| > |-.47943|$

$|-.92106 - .092106| > |-.47943|$

$\therefore$ at $x = -.3$ and $x = -.5$ the curve lies above the tangent to the curve at $Q(-.4, \sin(-.4))$



tan line:

$y = \sin(-.4) + \cos(-.4)(x + .4)$

$= -\sin .4 + \cos .4(x + .4)$

$= -.38942 + .92106(x + .4)$

In the linear substitution discussion of Section 4-3 we develop a specific instance of the "chain rule" as we need it to study a particular function. This is typical of our approach and reflects our concurrence with the following statements from the C.E.E.B. Calculus AB Syllabus:

> "This course is intended for students who have a thorough knowledge of college preparatory mathematics... It does not assume that they have acquired sound understanding of the theory of elementary functions. The development of this understanding has first priority.
>
> "A course in elementary functions and introductory calculus can be arranged in many ways ... In this version the special functions are first studied in some detail with the aid of the calculus, which is introduced intuitively, and later [Chapter 8] the general techniques of calculus are developed and applied to a wide class of functions."

As an alternative to the difference quotient approach used in Section 4-3 (but still quite within the spirit of the foregoing paragraphs), we could have obtained $D[k \sin (ax + b)]$ by the following method. Writing x as $c + (x - c)$, we substitute to get

$$\begin{aligned} k \sin (ax + b) &= k \sin [a(c + (x - c)) + b] \\ &= k \sin [(ac + b) + a(x - c)]. \end{aligned}$$

Using addition formula (6) of Section 3-5, we get $k \sin [(ac + b) + a(x - c)]$ $= k \sin (ac + b) \cos a(x - c) + k \cos (ac + b) \sin a(x - c)$. Then we argue that if x is so close to c that $|a(x - c)|$ is small, we can replace $\cos a(x - c)$ and $\sin a(x - c)$ by their best linear approximations: $\cos a(x - c)$ by one and $\sin a(x - c)$ by $a(x - c)$. As x approaches c,

$$k \sin (ac + b) \cos a(x - c) + k \cos (ac + b) \sin a(x - c)$$

approaches

$$[k \sin (ac + b)] \cdot (1) + [k \cos (ac + b)] \cdot (a(x - c)).$$

The equation of the tangent to the graph of $y = k \sin (ax + b)$ at the point where $x = c$ is $y = k \sin (ac + b) + [ka \cos (ac + b)](x - c)$. The slope of the tangent to the graph of

$$f : x \rightarrow k \sin (ax + b)$$

at the point $(c, f(c))$ is given by

$$f'(c) = ka \cos (ac + b).$$

Solutions Exercises 4-3

1. (a) $3\cos(3x - \frac{\pi}{4})$

   (b) $\sin(\frac{x}{2} + \frac{2\pi}{3})$

   (c) $\cos(\frac{\pi}{3} - x)$

   (d) $\frac{5}{3}\cos(\frac{\pi}{6} - \frac{x}{3})$

   (e) $-\frac{1}{2}\cos(-x + \frac{\pi}{2})$

   (f) $-2x\sin(2\pi - 2x)$

   or $2\pi\sin 2x$

2. (a) (i) $f(x) = \cos(-x + \frac{\pi}{3})$

   $f'(x) = \sin(-x) + \frac{\pi}{3})$

   $f'(-\frac{7\pi}{6}) = \sin(+\frac{7\pi}{6} + \frac{\pi}{3}) = \sin(\frac{3\pi}{2}) = -1$

   (ii) $f(x) = -\sin(2x - \frac{\pi}{6})$

   $f'(x) = -2\cos(2x - \frac{\pi}{6})$

   $f'(\frac{\pi}{12}) = -2\cos(\frac{\pi}{3} - \frac{\pi}{6}) = -2\cos\frac{\pi}{6} = -\sqrt{3}$

   (iii) $f(x) = 3\cos(\frac{\pi}{4} + 2x)$

   $f'(x) = -6\sin(\frac{\pi}{4} + 2x)$

   $f'(\frac{\pi}{2}) = -6\sin(\frac{\pi}{4} + \pi) = -6\sin\frac{5\pi}{4} = \frac{6}{\sqrt{2}}$.

   (iv) $f(x) = \frac{1}{2}\sin(\frac{3\pi}{2} - \frac{x}{2})$

   $f'(x) = -\frac{1}{4}\cos(\frac{3\pi}{2} - \frac{x}{2})$

   $f'(0) = -\frac{1}{4}\cos(\frac{3\pi}{2} - 0) = -\frac{1}{4}(0) = 0$

   (b)

	Point of tangency	Slope of tan line	Eqn. of tangent line
(i)	$(-\frac{7\pi}{6}, 0)$	-1	$y = 0 - 1(x + \frac{7\pi}{6})$ or $y = -(x + \frac{7\pi}{6})$
(ii)	$(\frac{\pi}{6}, -\frac{1}{2})$	$-\sqrt{3}$	$y = -\frac{1}{2} - \sqrt{3}(x - \frac{\pi}{6})$
(iii)	$(\frac{\pi}{2}, \frac{-3}{\sqrt{2}})$	$\frac{6}{\sqrt{2}}$	$y = \frac{-3}{\sqrt{2}} + \frac{6}{\sqrt{2}}(x - \frac{\pi}{2})$ or $y = \frac{-3}{2}\sqrt{2} + 3\sqrt{2}(x - \frac{\pi}{2})$
(iv)	$(0, -\frac{1}{2})$	0	$y = -\frac{1}{2} + 0(x - 0)$ or $y = -\frac{1}{2}$

3. (a) If $g(x) = cf(x)$, then $g'(x) = cf'(x)$

   (b) $f(x) = \sin 2x$

   $g(x) = -2 \cos (2x - \frac{\pi}{2}) = -2 \cos (\frac{\pi}{2} - 2x) = -2 \sin 2x$

   $\therefore \; g(x) = -2\, f(x)$

   $f'(x) = 2 \cos 2x$

   $g'(x) = 4 \sin (2x - \frac{\pi}{2}) = -4 \sin (\frac{\pi}{2} - 2x) = -4 \cos 2x$

   $= -2(2 \cos 2x)$

   $\therefore \; g'(x) = -2\, f'(x)$

4. (a) (i) $f(x) = \sin x + \cos x$

   $f'(x) = \cos x - \sin x$

   When the tangent is horizontal, $f' = 0$

   $\therefore \; \cos x = \sin x$ and $x = \frac{\pi}{4}, \frac{5\pi}{4}$.

   If the interval is not restricted to $0 \leq x < 2\pi$ the solution is $\frac{\pi}{4} + n\pi$ (n, integer).

   Alternate soln:

   $f(x) = \sqrt{2} (\sin x \cdot \frac{1}{\sqrt{2}} + \cos x \cdot \frac{1}{\sqrt{2}})$

   $= \sqrt{2} (\sin x \cos \frac{\pi}{4} + \cos x \sin \frac{\pi}{4})$

   $= \sqrt{2} \sin (x + \frac{\pi}{4})$

   $f'(x) = \sqrt{2} \cos (x + \frac{\pi}{4}) = 0$

   $\therefore \; x + \frac{\pi}{4} = \frac{\pi}{2}, \frac{3\pi}{2}$

   $\therefore \; x = \frac{\pi}{4}, \frac{5\pi}{4}$

   (See comment after first solution.)

   (ii) Equation of tangent at $(\frac{\pi}{4}, \sqrt{2})$: $y = \sqrt{2}$

   Equation of tangent at $(\frac{5\pi}{4}, -\sqrt{2})$: $y = -\sqrt{2}$

(b) (i) $g(x) = 4 \sin \frac{x}{2} + 4 \cos \frac{x}{2}$

$g'(x) = 2 \cos \frac{x}{2} - 2 \sin \frac{x}{2} = 0$

$\frac{x}{2} = \frac{\pi}{4}, \frac{5\pi}{4}$

$x = \frac{\pi}{2}, \frac{5\pi}{2}$

Since $\frac{5\pi}{2} > 2\pi$, $\underline{x = \frac{\pi}{2}}$.

The interval is not restricted to $0 \leq x < 2\pi$, the solution is $x = \frac{\pi}{2} + 2n\pi$ (n, integer).

Alternate solution:

$$g(x) = 4\sqrt{2}(\sin \frac{x}{2} \cos \frac{\pi}{4} + \cos \frac{x}{2} \sin \frac{\pi}{4})$$

$$= 4\sqrt{2} \sin (\frac{x}{2} + \frac{\pi}{4})$$

$$g'(x) = 2\sqrt{2} \cos (\frac{x}{2} + \frac{\pi}{4}) = 0$$

$$\therefore \frac{x}{2} + \frac{\pi}{4} = \frac{\pi}{2}$$

$$\frac{x}{2} = \frac{\pi}{4}$$

$$\underline{x = \frac{\pi}{2}}$$

(See comment after first solution.)

(ii) Equation of tangent at $(\frac{\pi}{2}, 4\sqrt{2})$: $y = 4\sqrt{2}$

(c) (i) $h(x) = 3 \sin (2x + \frac{\pi}{4}) + 3 \cos (2x + \frac{\pi}{4})$

$h'(x) = 6 \cos (2x + \frac{\pi}{4}) - 6 \sin (2x + \frac{\pi}{4}) = 0$

$\therefore 2x + \frac{\pi}{4} = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$

$2x = 0, \pi, 2\pi, 3\pi$

$x = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}$.

If the interval is not restricted to $0 \leq x < 2\pi$, the solution is $n\frac{\pi}{2}$ (n, integer.)

Alternate Solution:

$$h(x) = 3\sqrt{2}\sin\left[\left(2x + \frac{\pi}{4}\right) + \frac{\pi}{4}\right] = 3\sqrt{2}\sin\left(2x + \frac{\pi}{2}\right)$$

$$h'(x) = 6\sqrt{2}\cos\left(2x + \frac{\pi}{2}\right) = 0$$

$$\therefore\ 2x + \frac{\pi}{2} = \frac{\pi}{2},\ \frac{3\pi}{2},\ \frac{5\pi}{2},\ \frac{7\pi}{2}$$

$$x = 0,\ \frac{\pi}{2},\ \pi,\ \frac{3\pi}{2}$$

(See comment after first solution.)

(ii) Equation of tangent at $(0, 3\sqrt{2})$ and $(\pi, 3\sqrt{2})$: $y = 3\sqrt{2}$

Equation of tangent at $(\frac{\pi}{2}, -3\sqrt{2})$ and $(\frac{3\pi}{2}, -3\sqrt{2})$: $y = -3\sqrt{2}$

5. $$\begin{aligned} f(x) &= k\cos(ax + b) \\ &= k\cos[a(c + (x - c)) + b] \\ &= k\cos[(ac + b) + a(x - c)] \\ &= k\cos(ac + b)\cos a(x - c) - k\sin(ac + b)\sin a(x - c) \end{aligned}$$

If c is fixed and x is sufficiently close to c, then we can replace $\cos a(x - c)$ by 1, its best linear approximation and replace $\sin a(x - c)$ by $a(x - c)$, its best linear approximation.

$$\therefore\ f(x) = k\cos(ac + b)\cdot 1 - k\sin(ac + b)\cdot a(x - c)$$

Thus we have the tangent line to

$$f(x) = k\cos(ax + b)$$

at the point $(c, k\cos(ac + b))$:

$$y = k\cos(ac + b) - ka\sin(ac + b)\cdot(x - c)$$

This line has the slope $-ka\sin(ac + b)$; therefore, the derivative of $f : x \to k\cos(ax + b)$ is the function f' whose value at c is given by

$$f'(c) = -ka\sin(ac + b)$$

or

$$f'(x) = -ka\sin(ax + b).$$

6. $D\,k \cos(ax + b) = D\,k \sin\left(\frac{\pi}{2} - ax - b\right)$

$$= D\,{-k} \sin\left(ax + b - \frac{\pi}{2}\right)$$

$$= -ka \cos\left(ax + b - \frac{\pi}{2}\right)$$

$$= -ka \cos\left(\frac{\pi}{2} - ax + b\right)$$

$$= -ka \sin(ax + b)$$

7. (a) When x is in radians $\lim_{x \to 0} \frac{\sin x}{x} = 1.$

We must use this statement to evaluate

$$\lim_{\alpha^\circ \to 0} \frac{\sin \alpha^\circ}{\alpha^\circ}$$

Since α degrees equals $\frac{\pi}{180}\alpha$ radians then

$$\frac{\sin \alpha^\circ}{\alpha^\circ} = \frac{\sin \frac{\pi}{180}\alpha}{\alpha^\circ}$$

and multiplying by $\dfrac{\frac{\pi}{180}}{\frac{\pi}{180}}$ we have

$$\frac{\sin \frac{\pi}{180}\alpha}{\alpha^\circ} = \frac{\pi}{180} \frac{\sin \frac{\pi}{180}\alpha}{\frac{\pi}{180}\alpha}$$

Thus $\lim_{\alpha^\circ \to 0} \frac{\sin \alpha^\circ}{\alpha^\circ} = \lim_{\alpha \to 0} \frac{\pi}{180} \frac{\sin \frac{\pi}{180}\alpha}{\frac{\pi}{180}\alpha}$

$$= \frac{\pi}{180} \lim_{\alpha \to 0} \frac{\sin \frac{\pi}{180}\alpha}{\frac{\pi}{180}\alpha}$$

$$= \frac{\pi}{180} \cdot 1$$

$$= \frac{\pi}{180}$$

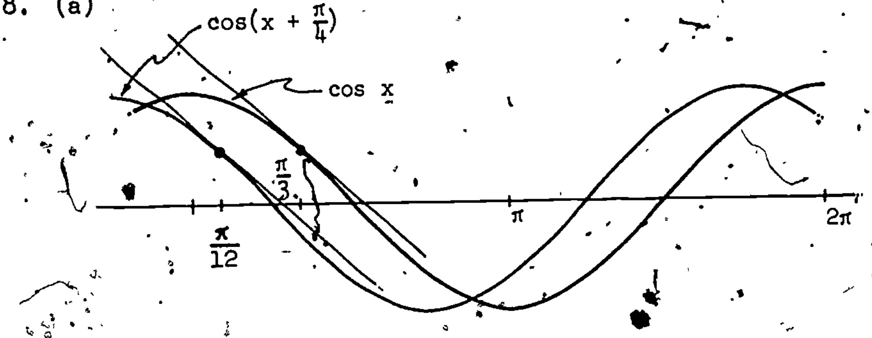
(b) $\cos x^\circ = \cos \frac{\pi}{180} x$

Thus $D \cos x^\circ = D \cos \frac{\pi}{180} x$

$$= -\frac{\pi}{180} \sin \frac{\pi}{180} x$$

or $= -\frac{\pi}{180} \sin x^\circ$

8. (a)



(b) $f \to \cos x, \quad f'(\frac{\pi}{3}) = \frac{-\sqrt{3}}{2}$

$g \to \cos(x + \frac{\pi}{3}), \quad g'(\frac{\pi}{12}) = \frac{-\sqrt{3}}{2}$

Thus the slopes of the tangent lines through $(\frac{\pi}{3}, f(\frac{\pi}{3}))$ and $(\frac{\pi}{12}, g(\frac{\pi}{12}))$ are the same and the tangent lines are parallel.

9. (a) $\lim\limits_{h \to 0} \frac{\sin(x + h) - \sin x}{h} = \cos x$

By definition of f' when $f : x \to \sin x$.

(b) $\lim\limits_{h \to 0} \frac{\cos x - \cos(x + h)}{h} = -\left[\lim\limits_{h \to 0} \frac{\cos(x + h) - \cos x}{h}\right]$

By definition of f' when

$f : x \to \cos x : \left[\lim\limits_{h \to 0} \frac{\cos(x + h) - \cos x}{h}\right] = -(-\sin x)$

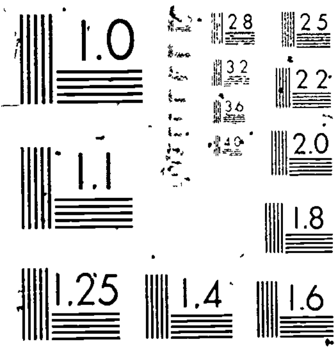
$= \sin x$

(c) $\lim\limits_{h \to 0} \frac{\cos(3x + h) - \cos 3x}{h} = -3 \sin 3x$

i.e., $f : x \to \cos 3x, \quad f' = -3 \sin 3x$

(d) $\lim\limits_{h \to 0} \frac{-3 \sin(\frac{x}{2} + \frac{\pi}{4} + h) + 3 \sin(\frac{x}{2} + \frac{\pi}{4})}{h} = -\frac{3}{2} \cos(\frac{x}{2} + \frac{\pi}{4})$

i.e., $f : x \to -3 \sin(\frac{x}{2} + \frac{\pi}{4})$; $f' : x \to -\frac{3}{2} \cos(\frac{x}{2} + \frac{\pi}{4})$



MICROCOPY RESOLUTION TEST CHART

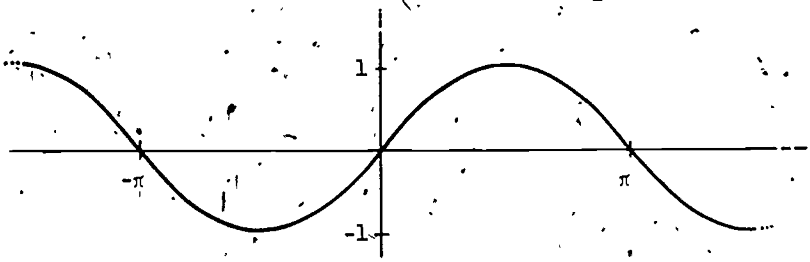
(e) $\lim\limits_{k \to 0} \dfrac{\sin(\pi + k) - \sin(\pi)}{k} = -1$

i.e., $f(x) = \sin x$; $f'(x) = \cos x$; $f'(\pi) = -1$

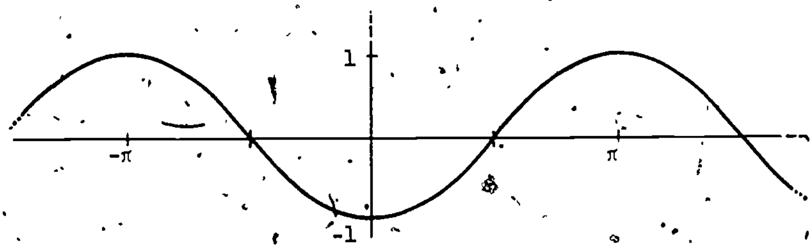
(f) $\lim\limits_{h \to 0} \dfrac{-\cos(\frac{\pi}{4} + h) + \cos\frac{\pi}{4}}{2h} = \lim\limits_{h \to 0} \left(-\frac{1}{2}\right)\left(\dfrac{\cos(\frac{\pi}{4} + h) - \cos\frac{\pi}{4}}{h}\right)$

i.e., $f(x) = -\frac{1}{2}\cos x$; $f'(x) = \frac{1}{2}\sin x$; $f'(\frac{\pi}{4}) = (\frac{1}{2})(\frac{\sqrt{2}}{2}) = \frac{\sqrt{2}}{4}$

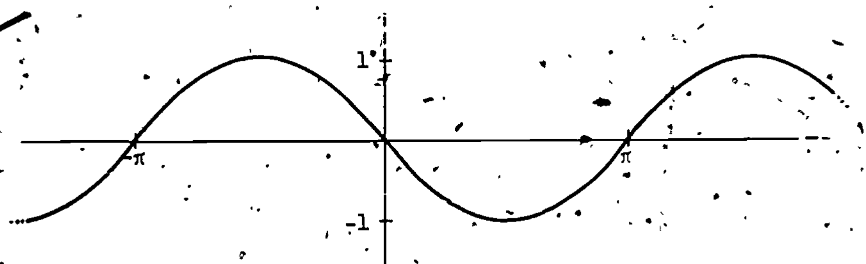
10. (a), (g), (i) $y = \sin x = \cos(x - \frac{\pi}{2}) = \cos(\frac{\pi}{2} - x)$



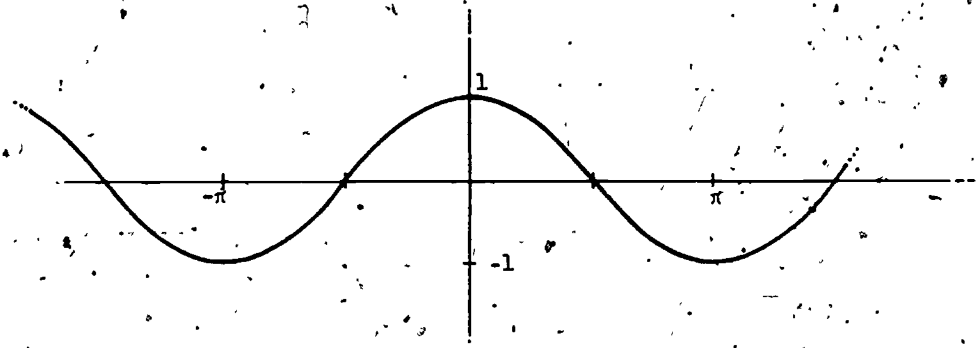
(b) $y = \sin(x - \frac{\pi}{2})$



(c), (d) $y = -\sin x = \sin(-x)$



(e), (f), (h) $\qquad y = \sin(\frac{\pi}{2} - x) = \cos x = \cos(-x)$



11. (a) $y = \sin x$

$\frac{dy}{dx} = \cos x$

(b) $y = \sin(x - \frac{\pi}{2})$

$\frac{dy}{dx} = \cos(x - \frac{\pi}{2}) = \sin x$

(c) $y = -\sin x$

$\frac{dy}{dx} = -\cos x$

(d) $y = \sin(-x)$

$\frac{dy}{dx} = -\cos(-x) = -\cos x$

(e) $y = \sin(\frac{\pi}{2} - x)$

$\frac{dy}{dx} = -\cos(\frac{\pi}{2} - x)$

$\frac{dy}{dx} = -\sin x$

(f) $y = \cos x$

$\frac{dy}{dx} = -\cos(\frac{\pi}{2} - x)$

(g) $y = \cos(\frac{\pi}{2} - x)$

$\frac{dy}{dx} = \sin(\frac{\pi}{2} - x) = \cos x$

(h) $y = \cos(-x)$

$\frac{dy}{dx} = \sin(-x) = -\sin x$

(i) $y = \cos(\frac{\pi}{2} - x)$

$\frac{dy}{dx} = \sin(\frac{\pi}{2} - x) = \cos x$

Solutions Exercises 4-4

1. $f : t \to r \cos \omega t$ and $g : t \quad r \sin \omega t$,

i.e., $t \to r \cos (\frac{s}{r})t$ and $g : t \to r \sin (\frac{s}{r})t$

If $s = \pi$,

<u>Horizontal</u>

position $f : t \to r \cos \frac{\pi}{r} t$

velocity $f : t \to -\pi \sin \frac{\pi}{r} t$

acceleration $f'' : t \to -\frac{\pi^2}{r} \cos \frac{\pi}{r} t$

<u>Vertical</u>

position $g : t \to r \sin \frac{\pi}{r} t$

velocity $g' : t \to \pi \cos \frac{\pi}{r} t$

acceleration $g'' : t \to \frac{\pi^2}{r} \sin \frac{\pi}{r} t$

(a) $r = 1$

<u>Horizontal</u>

position $f : t \to \cos \pi t$

velocity $f' : t \to -\pi \sin \pi t$

acceleration $f'' : t \to -x^2 \cos \pi t$

<u>Vertical</u>

position $g : \sin \pi t$

velocity $g' : \pi \cos \pi t$

acceleration $g'' : -\pi^2 \sin \pi t$

(i) If $t = 0$ $f(0) = \cos 0 = 1$ $\quad g(0) = \sin 0 = 0$

$f'(0) = -\pi \sin 0 = 0.$ $\quad g'(0) = \pi \cos 0 = \pi$

$f''(0) = -\pi^2 \cos 0 = -\pi^2$ $\quad g''(0) = -\pi^2 \sin 0 = 0$

(ii) If $t = \frac{1}{2}$ $f(\frac{1}{2}) = \cos \frac{\pi}{2} = 0$ $\quad g(\frac{1}{2}) = \sin \frac{\pi}{2} = 1$

$f'(\frac{1}{2}) = -\pi \sin \frac{\pi}{2} = -\pi$ $\quad g'(\frac{1}{2}) = \pi \cos \frac{\pi}{2} = 0$

$f''(\frac{1}{2}) = -\pi^2 \cos \frac{\pi}{2} = 0$ $\quad g''(\frac{1}{2}) = -\pi^2 \sin \frac{\pi}{2} = -\pi^2$

(iii) If $t = 1$ $f(1) = \cos \pi = -1$ $\quad g(1) = \sin \pi = 0$

$f'(1) = -\pi \sin \pi = 0$ $\quad g'(1) = \pi \cos \pi = -\pi$

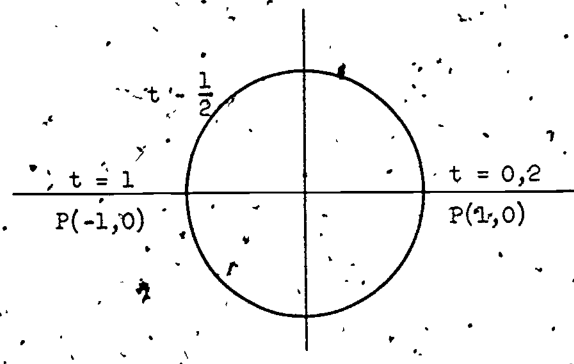
$f''(1) = -\pi^2 \cos \pi = \pi^2$ $\quad g''(1) = -\pi^2 \sin \pi = 0$

(iv) If $t = 2$ $f(2) = \cos 2\pi = 1$ $\quad g(2) = \sin 2\pi = 0$

$f'(2) = -\pi \sin 2\pi = 0$ $\quad g'(2) = \pi \cos 2\pi = \pi$

$f''(2) = -\pi^2 \cos 2\pi = -\pi^2$ $\quad g''(2) = -\pi^2 \sin 2\pi = 0$

Note that in the unit circle P has returned to its starting point in 2 seconds.



(b) $r = 2$

Horizontal	Vertical
position $f : t \to 2 \cos \frac{\pi}{2} t$	position $g : t \to 2 \sin \frac{\pi}{2} t$
velocity $f' : t \to -\pi \sin \frac{\pi}{2} t$	velocity $g' : t \to \pi \cos \frac{\pi}{2} t$
acceleration $f'' : t \to -\frac{\pi^2}{2} \cos \frac{\pi}{2} t$	acceleration $g'' : t \to -\frac{\pi^2}{2} \sin \frac{\pi}{2} t$

(i) If $t = 0$

$f(0) = 2 \cos 0 = 2$ $\qquad$ $g(0) = 2 \sin 0 = 0$

$f'(0) = -\pi \sin 0 = 0$ $\qquad$ $g'(0) = \pi \cos 0 = \pi$

$f''(0) = -\frac{\pi^2}{2} \cos 0 = -\frac{\pi}{2}$ $\qquad$ $g''(0) = -\frac{\pi^2}{2} \sin 0 = 0$

(ii) If $t = 1$

$f(1) = 2 \cos \frac{\pi}{2} = 0$ $\qquad$ $g(1) = 2 \sin \frac{\pi}{2} = 2$

$f'(1) = -\pi \sin \frac{\pi}{2} = -\pi$ $\qquad$ $g'(1) = \pi \cos \frac{\pi}{2} = 0$

$f''(1) = -\frac{\pi^2}{2} \cos \frac{\pi}{2} = 0$ $\qquad$ $g''(1) = -\frac{\pi^2}{2} \sin \frac{\pi}{2} = -\frac{\pi^2}{2}$

(iii) If $t = 2$

$f(2) = 2 \cos \pi = -2$ $\qquad$ $g(2) = 2 \sin \pi = 0$

$f'(2) = -\pi \sin \pi = 0$ $\qquad$ $g'(2) = \pi \cos \pi = -\pi$

$f''(2) = -\frac{\pi^2}{2} \cos \pi = \frac{\pi^2}{2}$ $\qquad$ $g''(2) = -\frac{\pi^2}{2} \sin \pi = 0$

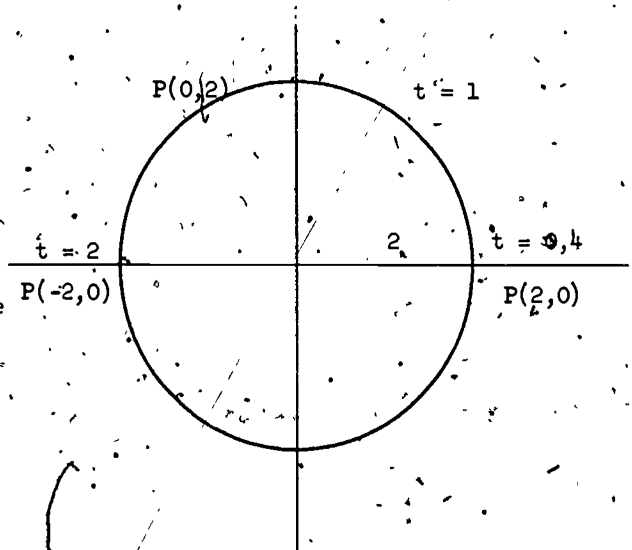
(iv) If $t = 4$,

$f(4) = 2 \cos 2\pi = 2$ $\qquad$ $g(4) = 2 \sin 2\pi = 0$

$f'(4) = -\pi \sin 2\pi = 0$ $\qquad$ $g'(4) = 2 \cos 2\pi = 0$

$f''(4) = -\frac{\pi^2}{2} \cos 2\pi = -\frac{\pi^2}{2}$ $\qquad$ $g''(4) = -\frac{\pi^2}{2} \sin 2\pi = 0$

Note that while P travels completely around the unit circle in 2 seconds, it travels exactly half the distance around the circle of radius of 2 in the same time.



(c) $r = 6$

Horizontal

position $f : t \to 6 \cos\left(\frac{\pi}{6}\right)t$

velocity $f' : t \to -\pi \sin\left(\frac{\pi}{6}\right)t$

acceleration $f'' : t \to -\frac{\pi^2}{6} \cos\left(\frac{\pi}{6}\right)t$

Vertical

position $g : t \to 6 \sin\left(\frac{\pi}{6}\right)t$

velocity $g' : t \to \pi \cos\left(\frac{\pi}{6}\right)t$

acceleration $g'' : t \to -\frac{\pi^2}{6} \sin\left(\frac{\pi}{6}\right)t$

(i) If $t = 0$, $f(0) = 6 \cos 0 = 6$ $g(0) = 6 \sin 0 = 0$

$f'(0) = -\pi \sin 0 = 0$ $g'(0) = \pi \cos 0 = \pi$

$f''(0) = -\frac{\pi^2}{6} \cos 0 = -\frac{\pi^2}{6}$ $g''(0) = -\frac{\pi^2}{6} \sin 0 = 0$

(ii) If $t = 1$, $f(1) = 6 \cos \frac{\pi}{6} = 3\sqrt{3}$ $g(1) = 6 \sin \frac{\pi}{6} = 3$

$f'(1) = -\pi \sin \frac{\pi}{6} = -\frac{\pi}{2}$ $g'(1) = \pi \cos \frac{\pi}{6} = \frac{\pi\sqrt{3}}{2}$

$f''(1) = -\frac{\pi^2}{6} \cos \frac{\pi}{6} = -\frac{\pi^2\sqrt{3}}{12}$ $g''(1) = -\frac{\pi^2}{6} \sin \frac{\pi}{6} = -\frac{\pi^2}{12}$

(iii) If $t = 2$, $f(2) = 6 \cos \frac{\pi}{3} = 3$ $g(2) = 6 \sin \frac{\pi}{2} = 3\sqrt{3}$

$f'(2) = -\pi \sin \frac{\pi}{3} = -\frac{\pi\sqrt{3}}{2}$ $g'(2) = \pi \cos \frac{\pi}{2} = \frac{\pi}{2}$

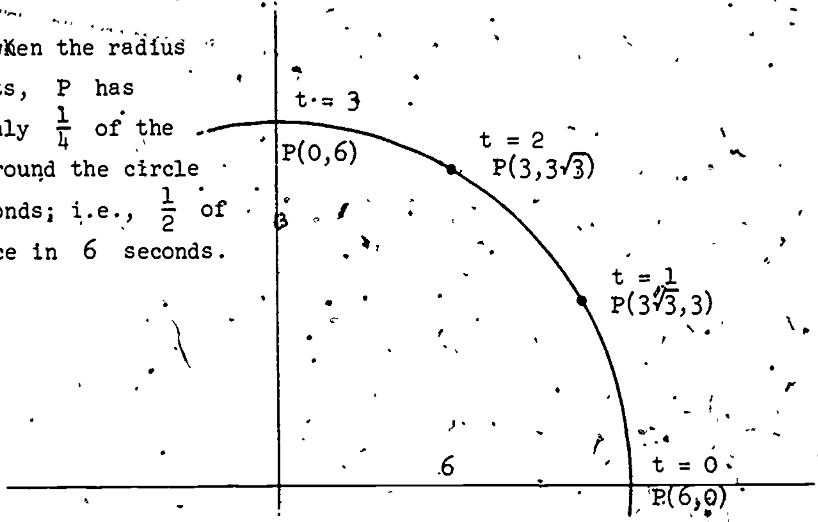
$f''(2) = -\frac{\pi^2}{6} \cos \frac{\pi}{3} = -\frac{\pi^2}{12}$ $g''(2) = -\frac{\pi^2}{6} \sin \frac{\pi}{3} = -\frac{\pi^2\sqrt{3}}{12}$

(iv) If $t = 3$,
$f(3) = 6 \cos \frac{\pi}{2} = 0$ $\qquad$ $g(3) = 6 \sin \frac{\pi}{2} = 6$

$f'(3) = -\pi \sin \frac{\pi}{2} = -\pi$ $\qquad$ $g'(3) = \pi \cos \frac{\pi}{2} = 0$

$f''(3) = -\frac{\pi^2}{6} \cos \frac{\pi}{2} = 0$ $\qquad$ $g''(3) = -\frac{\pi^2}{6} \sin \frac{\pi}{2} = -\frac{\pi^2}{6}$

Note that when the radius is 6 units, P has traveled only $\frac{1}{4}$ of the distance around the circle in 3 seconds; i.e., $\frac{1}{2}$ of the distance in 6 seconds.



2. The position in all 3 cases is the same relative to the circle; i.e., (0,1), (0,2) and (0,6), respectively. The velocity in all 3 cases is exactly the same;

i.e., if $r = 1$, $f'(\frac{1}{2}) = -\pi$ and $g'(\frac{1}{2}) = 0$;

if $r = 2$, $f'(1) = -\pi$ and $g'(1) = 0$;

if $r = 6$, $f'(3) = -\pi$ and $g'(3) = 0$.

The horizontal acceleration is zero, but as the radius of the circle increases the vertical acceleration becomes less negative;

i.e., if $r = 1$, $f''(\frac{1}{2}) = 0$ and $g''(\frac{1}{2}) = -\pi^2$

if $r = 2$, $f''(1) = 0$ and $g''(1) = -\frac{\pi^2}{2}$

if $r = 6$, $f''(3) = 0$ and $g''(3) = -\frac{\pi^2}{6}$

3. The position in all cases is the same relative to the circle; i.e., $(1,0)$, $(2,0)$, and $(6,0)$ respectively.

The velocity for $r = 1$, 2, and 6 is the same for $t = 0$; i.e., $f'(0) = 0$ and $g'(0) = \pi$. Also

when $r = 1$ $f'(2) = 0$ and $g'(2) = \pi$;
when $r = 3$ $f'(4) = 0$ and $g'(4) = \pi$, and conjecturing,
when $r = 6$ $f'(12) = 0$ and $g'(12) = \pi$

The vertical acceleration is zero, but as the radius of the circle increases, the horizontal acceleration becomes less negative; i.e., if

$$r = 1 \quad f''(2) = -\pi^2 \quad \text{and} \quad g''(2) = 0$$

$$r = 3 \quad f''(4) = \frac{\pi^2}{2} \quad \text{and} \quad g''(4) = 0$$

$$r = 6 \quad f''(12) = -\frac{\pi^2}{6} \quad \text{and} \quad g''(12) = 0$$

4. $f : t \to r \cos \omega t$ $\qquad$ $g : t \to r \sin \omega t$

$f' : t \to -r \sin \omega t$ $\qquad$ $g' : t \to r \cos \omega t$

$f'' : t \to -r\omega^2 \cos \omega t$ $\qquad$ $g'' : t \to -r\omega^2 \sin \omega t$

$$\sqrt{(f'')^2 + (g'')^2} = \sqrt{r^2\omega^4(\cos^2 \omega t + \sin^2 \omega t)} = r\omega^2 = r\left(\frac{s}{r}\right)^2 = \frac{s^2}{r} = s \cdot \omega$$

i.e., the square root of the sum of the squares of the acceleration components is the product of the speed and the angular velocity.

5. (a) Since $\qquad f(0) = 3$

and $\qquad f(0) = A \cos 0 = A,$

$$A = 3$$

(b) The period T satisfies

$$\sqrt{\frac{k}{m}}\, T = 2\pi.$$

Hence, $\qquad T = 2\pi \sqrt{\frac{m}{k}}.$

(c) $f'(t) = -A \sqrt{\frac{k}{m}} \sin \sqrt{\frac{k}{m}}\, t.$ Hence, $f'(0) = 0.$

(d) $f''(t) = -\frac{Ak}{m} \cos \sqrt{\frac{k}{m}}\, t.$ $f''(0) = -\frac{Ak}{m}.$

The velocity is decreasing (a change from positive through 0 to negative is a decrease).

(e) $f''(t) = -\frac{k}{m} A \cos \sqrt{\frac{k}{m}}\, t = -\frac{k}{m} f(t).$ We abbreviate: $f'' = -\frac{k}{m} f.$

(f) 3 since $3 \cos \sqrt{\frac{k}{m}}\, t$ has the minimum value $3(-1) = -3$.

(g) $f(t) = 3 \cos \sqrt{\frac{k}{m}}\, t$

$f(\frac{\pi}{2} \sqrt{\frac{m}{k}}) = 3 \cos \frac{\pi}{2} = 0.$

$f'(t) = -3 \sqrt{\frac{k}{m}} \sin \sqrt{\frac{k}{m}}\, t.$

$f'(\frac{\pi}{2} \sqrt{\frac{m}{k}}) = -3 \sqrt{\frac{k}{m}} \sin \frac{\pi}{2} = -3 \sqrt{\frac{k}{m}}.$

$f''(t) = -3 \frac{k}{m} \cos \sqrt{\frac{k}{m}}\, t.$

$f''(\frac{\pi}{2} \sqrt{\frac{m}{k}}) = -3 \frac{k}{m} \cos \frac{\pi}{2} = 0.$

The mass is moving to the left at this time.

(h) $f(t) = 0$ when $\sqrt{\frac{k}{m}}\, t = \frac{3\pi}{2}$, that is, when $t = \frac{3\pi}{2} \sqrt{\frac{m}{k}}$.

$f'(\frac{3\pi}{2} \sqrt{\frac{m}{k}}) = -3 \sqrt{\frac{k}{m}} \sin \frac{3\pi}{2} = 3 \sqrt{\frac{k}{m}}$

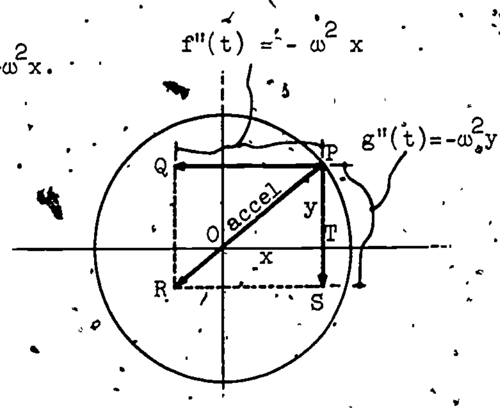
$f''(\frac{3\pi}{2} \sqrt{\frac{m}{k}}) = -3 \frac{k}{m} \cos \frac{3\pi}{2} = 0.$

The mass is moving to the right at this time.

6. $x = f(t) = A \cos \omega t,$

$f'(t) = -A\omega \sin \omega t,$

$f''(t) = -A\omega^2 \cos \omega t = -\omega^2 f(t) = -\omega^2 x.$



Similarly

$y = g(t) = A \sin \omega t,$

$g'(t) = A\omega \cos \omega t,$

$g''(t) = -A\omega^2 \sin \omega t = -\omega^2 y.$

$QP = RS = \omega^2 x$

$SP = \omega^2 y$

Triangles RSP and OTP are similar. PR passes through O. (If ω were less than 1 it would not reach this far.)

Solutions Exercises 4-5

1. $f : x \to \sin x$
   $f' : x \to \cos x$
   $f'' : x \to -\sin x$
   $f''' : x \to -\cos x$
   $f^{(4)} : x \to \sin x$

   $g : x \to \cos x$
   $g' : x \to -\sin x$
   $g'' : x \to -\cos x$
   $g''' : x \to \sin x$
   $g^{(4)} : x \to \cos x$

   (a) $f^{(21)} x \to \cos x$
   $f^{(18)} x \to -\sin x$
   $f^{(35)} x \to -\cos x$
   $f^{(16)} x \to \sin x$

   (b) $g^{(31)} x \to \sin x$
   $g^{(42)} x \to -\cos x$
   $g^{(20)} x \to \cos x$
   $g^{(101)} x \to -\sin x$

2. (a) $f(x) = \sin x$
   (i) $f'(\frac{\pi}{6}) = \frac{\sqrt{3}}{2}$
   (ii) $f''(\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$
   (iii) $f'''(\frac{\pi}{4}) = -\frac{1}{\sqrt{2}}$
   (iv) $f^{iv}(\frac{5\pi}{6}) = \frac{1}{2}$

   (b) $g(x) = \cos x$
   (i) $g'(\frac{\pi}{6}) = \frac{1}{2}$
   (ii) $g''(\frac{3\pi}{4}) = \frac{1}{\sqrt{2}}$
   (iii) $g'''(\frac{2\pi}{3}) = \frac{\sqrt{3}}{2}$
   (iv) $g^{iv}(\frac{5\pi}{6}) = -\frac{\sqrt{3}}{2}$

3. (a) $f(x) = A \sin ax$
   (i) $f'(x) = Aa \cos ax$
   (ii) $f''(x) = -Aa^2 \sin ax$
   (iii) $f'''(x) = -Aa^3 \cos ax$
   (iv) $f^{iv}(x) = Aa^4 \sin ax$

   (b) $f(x) = B \cos bx$
   (i) $g'(x) = -Bb \sin bx$
   (ii) $g''(x) = -Bb^2 \cos bx$
   (iii) $g'''(x) = Bb^3 \sin bx$
   (iv) $g^{iv}(x) = Bb^3 \cos bx$

4. (a) $f(x) = 3 \sin \pi x$
   $f'(x) = 3\pi \cos \pi x$
   $f''(x) = -3\pi^2 \sin \pi x$
   $f'''(x) = -3\pi^3 \cos \pi x$
   $f^{(4)}(x) = 3\pi^4 \sin \pi x$

   (b) $g(x) = 2 \cos \frac{x}{2}$
   $g'(x) = -\sin \frac{x}{2}$
   $g''(x) = -\frac{1}{2} \cos \frac{x}{2}$
   $g'''(x) = \frac{1}{4} \sin \frac{x}{2}$
   $g^{(4)}(x) = \frac{1}{8} \cos \frac{x}{2}$

5. (a) $f(x) = 3 \sin (\frac{x}{2} + \frac{\pi}{4})$; $f(0) = 3 \sin (\frac{\pi}{4}) = \frac{3}{\sqrt{2}}$

(b) (i) $f'(x) = \frac{3}{2} \cos (\frac{x}{2} + \frac{\pi}{4})$

(ii) $f'(\pi) = \frac{3}{2} \sin (\frac{3\pi}{4}) = \frac{3}{2} \cdot \frac{1}{\sqrt{2}} = \frac{3\sqrt{2}}{4}$

(c) (i) $f''(x) = - \frac{3}{4} \sin (\frac{x}{2} + \frac{\pi}{4})$

(ii) $f''(\frac{\pi}{2}) = - \frac{5}{4} \sin (\frac{\pi}{2}) = - \frac{3}{4} \cdot 1 = - \frac{3}{4}$

(d) (i) $f'''(x) = - \frac{3}{8} \cos (\frac{x}{2} + \frac{\pi}{4})$

(ii) $f'''(- \frac{3\pi}{2}) = - \frac{3}{8} \cos (- \frac{\pi}{2}) = - \frac{3}{8}(0) = 0$

(e) (i) $f^{(4)}(x) = \frac{3}{16} \sin (\frac{x}{2} + \frac{\pi}{4})$

(ii) $f^{(4)}(2\pi) = \frac{3}{16} \sin (\frac{5\pi}{4}) = \frac{3}{16} \cdot \frac{1}{\sqrt{2}} = \frac{3\sqrt{2}}{32}$

6. (a) $g(x) = -2 \cos (2x + \frac{\pi}{2})$; $g(0) = -2 \cos (\frac{\pi}{2}) = -2 \cdot 0 = 0$

(b) (i) $g'(x) = 2^2 \sin (2x + \frac{\pi}{2}$

(ii) $g'(\pi) = 2^2 \sin (\pi) = 4 \cdot 0 = 0$

(c) (i) $g''(x) = 2^3 \cos (2x + \frac{\pi}{2})$

(ii) $g''(- \frac{\pi}{12}) = 2^3 \cos (\frac{\pi}{3}) = 2^3(\frac{1}{2}) = 4$

(d) (i) $g'''(x) = -2^4 \sin (2x + \frac{\pi}{2})$

(ii) $g'''(0) = -2^4 \sin (\frac{\pi}{2}) = -2^4(1) = -16$

(e) (i) $g^{(4)}(x) = -2^5 \cos (2x + \frac{\pi}{2})$

(ii) $g^{(4)}(\frac{\pi}{2}) = -2^5 \cos (\frac{3\pi}{2}) = -2^5(-1) = 32$

7. Let $f : x \rightarrow \sin x$ and write $n = 4k + \epsilon$, $0 \leq \epsilon \leq 3$. The four cases are then

$$f^{(4k)} : x \rightarrow \sin x$$
$$f^{(4k+1)} : x \rightarrow \cos x$$
$$f^{(4k+2)} : x \rightarrow -\sin x$$
$$f^{(4k+3)} : x \rightarrow -\cos x$$

(Special cases are given in the answer to Exercise 1).

8. Prove that

$$\cos x \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$f(x) = \cos x = f^{4I}(x)$ $\qquad$ $f(0) = f^{4I}(0) = 1 = p(0)$

$f'(x) = -\sin x = f^{4I+1}(x)$ $\qquad$ $f'(0) = f^{4I+1}(0) = 0 = p'(0) = p^5(0) = \cdots$

$f''(x) = -\cos x = f^{4I+2}(x)$ $\qquad$ $f''(0) = f^{4I+2}(0) = -1 = p''(0) = p^6(0) = \cdots$

$f'''(x) = \sin x = f^{4I+3}(x)$ $\qquad$ $f'''(0) = f^{4I+3}(0) = 0 = p'''(0) = p^7(0) = \cdots$

$f^{(4)}(x) = \cos x = f^{4I}(x)$ $\qquad$ $f^{(4)}(0) = f^{4I}(0) = 1 = p^4(0) = p^8(0) = \cdots$

where I = integer.

$p(x) = a_0 + c_1x + a_2x^2 + a_3x^3 + a_4x^4 + \cdots,$ $\qquad$ $p(0) = 0!a_0 = 1$

$p'(x) = a_1 + 2a_2x + 3a_3x^2 + 4a_4x^3 + 5a_5x^4 + \cdots$ $\qquad$ $p'(0) = 1!a_1 = 0$

$p''(x) = 2a_2 + 3!a_3x + 4 \cdot 3a_4x^2 + 5 \cdot 4a_3x^3 + \cdots$ $\qquad$ $p''(0) = 2!a_2 = -1$

$p'''(x) = 3!a_3 + 4!a_4x + 5 \cdot 4 \cdot 3a_3x^2 + \cdots$ $\qquad$ $p'''(0) = 3!a_3 = 0$

$p^{(4)}(x) = 4!a_4 + 5!a_3x + \frac{6!}{2!}a_4x^2 + \frac{7!}{3!}a_5x^3 + \cdots$ $\qquad$ $p^{(4)}(0) = 4!a_4 = 1$

$p^{(5)}(x) = 5!a_3 + 6!a_4x + \frac{7!}{2!}a_5x^2 + \cdots$ $\qquad$ $p^{(5)}(0) = 5!a_5 = 0$

....

$\therefore\ a_0 = 1 \quad a_2 = -\frac{1}{2}! \quad a_4 = \frac{1}{4}! \quad a_6 = -\frac{1}{6}! \ \ldots\ldots$

$\quad\ a_1 = 0 \quad a_3 = 0 \quad a_5 = 0 \quad a_7 = 0 \ \ldots\ldots$

$\therefore\ p(x) \doteq 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + (-1)^k\frac{x^n}{n!}$

i.e., $\cos x \approx 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + (-1)^k\frac{x^n}{n!}$

where n is even $k = \frac{n}{2}$.

9. $\sin x = x - \frac{x^3}{3!} + R$

(a) $\sin(0.2) \approx (0.2) - \frac{(0.02)^3}{3 \cdot 2 \cdot 1}$

$\approx 0.2 - \frac{.008}{6}$

$\approx 0.2 - 0.00133$

≈ 0.19867

(correct to 5 place accuracy.)

(b) $|R| < \frac{(0.2)^5}{5!}$

$< \frac{.00032}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$

$< \frac{.00032}{120}$

$< .000003$

10. $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} R$

(a) $\cos(0.2) \approx 1 - \frac{(0.2)^2}{2\cdot 1} + \frac{(0.2)^2}{4\cdot 3\cdot 2\cdot 1}$

$\approx 1 - \frac{0.04}{2} + \frac{0.0016}{24}$

$\approx 1 - 0.02 + .00007$

$\therefore \cos(0.2) \approx 0.98007$ with

$|R| < .000001$

(correct to 5 place accuracy.)

(b) $|R| < \frac{(.2)^6}{6}$

$< \frac{.000064}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$

$< \frac{.000064}{720}$

11. (a) (i) $\cos\frac{1}{2} \approx 1 - (\frac{1}{2})^2 \frac{1}{2\cdot 1} = 1 - \frac{1}{8} = \frac{7}{8} = 0.875$

(ii) $\cos\frac{1}{2} \approx 1 - (\frac{1}{2})^2 \frac{1}{2\cdot 1} + (\frac{1}{2})^4 \frac{1}{4\cdot 3\cdot 2\cdot 1}$

$= 1 - \frac{1}{8} + \frac{1}{1536}$

(iii) $\cos\frac{1}{2} \approx 1 - (\frac{1}{2})^2 \frac{1}{2\cdot 1} + (\frac{1}{2})^4 \frac{1}{4\cdot 3\cdot 2\cdot 1} - (\frac{1}{2})^6 \frac{1}{6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$

$= 1 - \frac{1}{8} + \frac{1}{1536} - \frac{1}{184320}$

(b) (i) $R_2 < \frac{1}{1536} \approx 0.000651$

Hence $\cos\frac{1}{2} \approx 0.88$ to 2-place accuracy.

(ii) $R_4 < \frac{1}{184320} \approx 0.0000054$

$\cos\frac{1}{2} \approx 0.87565$ to 5-place accuracy.

(iii) $R_6 < (\frac{1}{2})^8 \frac{1}{8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}$

$= \frac{1}{224}\cdot\frac{1}{184320} \approx 0.00000002$

8-place accuracy.

12. Suppose $f : x \to \sin x$ and $f(x) = p(x)$, $a \le x \le b$, where p is a polynomial of degree n. Then

$$f^{(n+1)}(x) \text{ is } \pm\sin x \text{ or } \pm\cos x$$

depending upon n. Thus we cannot have

$$f^{(n+1)} = p^{(n+1)}$$

since $p^{(n+1)}$ is the zero function.

13. (a) $\sin x = x - \frac{x^3}{6} + \frac{x^5}{120} - R_5$ where $0 < R_5 < \frac{x^7}{7!}$

Then $\sin x - (x - \frac{x^3}{6}) = \frac{x^5}{120} - R_5$

and

$$\frac{\sin x - (x - \frac{x^3}{6})}{x^5} = \frac{1}{120} - \frac{R_5}{x^5},$$

$$0 < \frac{R_5}{x^5} < \frac{x^2}{7!}$$

$\therefore$ Limit $= \frac{1}{120}$

This process can be abbreviated as follows:

$$\sin x \approx x - \frac{x^3}{6} + \frac{x^5}{120}.$$

$$\sin x - (x - \frac{x^3}{6}) \approx \frac{x^5}{120}$$

$$\frac{\sin x - (x - \frac{x^3}{6})}{x^5} \approx \frac{1}{120}$$

Hence, Limit $= \frac{1}{120}$

(b) $\lim\limits_{x \to 0} \frac{\sin x^2 - (x^2 - \frac{x^6}{6})}{x^{10}} = \lim\limits_{t \to 0} \frac{\sin t - (t - \frac{t^3}{6})}{t^5} = \frac{1}{120}$

(see part (a))

(c) $\lim\limits_{t \to 0} \frac{t - \sin t}{\frac{t^3}{8}}$

$\lim\limits_{t \to 0} 8(\frac{t - \sin t}{t^3}) = \frac{8}{6} = \frac{4}{3}$, since $\sin t \approx t - \frac{t^3}{6}$

and $\frac{t - \sin t}{t^3} \approx \frac{1}{6}$.

(d) $\lim_{t \to 0} \frac{t - \sin t}{2t}$

$\sin t \approx t - \frac{t^3}{6}$

$t - \sin t \approx \frac{t^3}{6}$

$\frac{t - \sin t}{2t} \approx \frac{t^3}{2t} = \frac{t^2}{2}$

Hence, Limit = 0.

(e). Let $x^2 = t$

$\lim_{x \to 0} \frac{1 - \cos x^2}{x^4} = \lim_{t \to 0} \frac{1 - \cos t}{t^2}$

Since $\cos t \approx 1 - \frac{t^2}{2}$

$1 - \cos t \approx \frac{t^2}{2}$

$\frac{1 - \cos t}{t^2} \approx \frac{1}{2}$

The required limit $= \frac{1}{2}$.

(f) Let $x^3 = t$

$\lim_{x \to 0} \frac{1 - \cos x^3}{x^4} = \lim_{t \to 0} \frac{1 - \cos t}{t^{4/3}}$

$1 - \cos t \approx \frac{t^2}{2}$

$\frac{1 - \cos t}{t^{4/3}} \approx \frac{1}{2} t^{2/3}$

Hence, the required limit is 0.

14. (a) $\sin x \approx x$

$\cos x \approx 1$

$\therefore \frac{\sin x}{\cos x} \approx \frac{x}{1} = x$ and $\lim_{x \to 0} \frac{\sin x}{\cos x} = 0.$

More precisely, if $x > 0$

$$\sin x = x - R_1 \qquad 0 < R_1 < \frac{x^3}{3!}$$

$$\cos x = 1 - R_2 \qquad 0 < R_2 < \frac{x^2}{2!}$$

$$\frac{\sin x}{\cos x} = \frac{x - R_1}{1 - R_2} = x\,\frac{(1 - \frac{R_1}{x})}{1 - R_2}$$

Since $\frac{R_1}{x}$ and R_2 approach 0 $\frac{1 - \frac{R_1}{x}}{1 - R_2}$ approaches 1.

Since x approaches 0 we obtain the same conclusion.

(b) The easiest solution is to write

$$\frac{x \cos x - \sin x}{x} = \cos x - \frac{\sin x}{x}$$

and to observe that both $\cos x$ and $\frac{\sin x}{x}$ approach 1. Hence, the required limit is 0. Otherwise,

$$\frac{x \cos x - \sin x}{x} \approx \frac{x(1 - \frac{x^2}{2}) - x}{x}$$

$$\approx \frac{x - \frac{x^2}{x} - x}{x}$$

$$= -\frac{x}{2}$$

Hence, the Limit = 0.

(c) $\sin x \approx x$

$\frac{\sin x}{\sqrt{x}} \approx \sqrt{x}$

Hence, $\frac{\sin x}{\sqrt{x}}$ approaches 0 as x approaches zero from the right. (Approach from the left is impossible since $\sqrt{x}$ is not defined for $x < 0$.)

15. (a) $\sin x \approx x$

$1 + \cos x \approx 2$

$\therefore \dfrac{\sin x}{1 + \cos x} \approx \dfrac{x}{2}$

Hence $\lim\limits_{x \to 0} \dfrac{\sin x}{1 + \cos x} = 0.$

One can also write

$$\frac{\sin x}{1 + \cos x} = \frac{\sin x\,(1 - \cos x)}{(1 + \cos x)(1 - \cos x)}$$

$$= \frac{\sin x\,(1 - \cos x)}{1 - \cos^2 x}$$

$$= \frac{\sin x\,(1 - \cos x)}{\sin^2 x}$$

$$= \frac{1 - \cos x}{\sin x}$$

By Example 4-5d, the limit is 0.

(b) $\sin x \approx x$

$\sin^2 x \approx x^2$

$1 - \cos x \approx \dfrac{x^2}{2}$

Hence, $\dfrac{\sin^2 x}{1 - \cos x} \approx \dfrac{x^2}{\frac{x^2}{2}} = 2$ and the required limit is 2.

It is better, however, to observe that

$$\sin^2 x = 1 - \cos^2 x = (1 - \cos x)(1 + \cos x)$$

and therefore

$$\frac{\sin^2 x}{1 - \cos x} = 1 + \cos x ,$$

which clearly approaches 2 as x approaches 0.

(c) $\sin t \approx t$

Hence $\sin x^2 \approx x^2$

$$x^2 \sin x^2 \approx x^4$$

$$1 - \cos x \approx \frac{x^2}{2}$$

$$(1 - \cos x)^2 \approx \frac{x^4}{4}$$

Then $$\frac{x^2 \sin x^2}{(1 - \cos x)^2} \approx \frac{x^4}{\frac{x^4}{4}} = 4$$

so that the required limit is 4.

16. Suppose $f : x \to \sin 2x$ and $g : x \to \sin x$. Note that

$$f^{(n)}(x) = 2^n g^{(n)}(2x)$$

so that

$$f^{(n)}(0) = 2^n g^{(n)}(0).$$

If $p(x) = a_0 + a_1 x + \ldots$ is the Taylor approximation to f at $x = 0$ and

$$q(x) = x - \frac{x^3}{3!} + \ldots$$

is the Taylor approximation to g at $x = 0$ then

$$p^{(k)}(0) = f^{(k)}(0) = 2^k g^{(k)}(0) = 2^k \cdot q^{(k)}(0).$$

$$\therefore \quad p(x) = 2x - \frac{2^3 x^3}{3!} + \ldots$$
$$= q(2x).$$

17. (a) Express p in powers of $x - a$:

$$p(x) = b_0 + b_1(x - a) + b_2(x - a)^2 + \ldots + b_n(x - a)^n.$$

Then

$$b_0 = p(a), \quad b_1 = p'(a), \quad b_2 = \frac{p''(a)}{2!}, \quad \ldots, \quad b_n = \frac{p^{(n)}(a)}{n!}$$

as shown in Section 2-13.

$\therefore$ The assumption $p(a) = f(a)$ guarantees that $b_0 = f(a)$, while the assumption (iii) guarantees that

$$b_1 = f'(a), \quad b_2 = \frac{f''(a)}{2!}, \quad \ldots, \quad b_n = \frac{f^{(n)}(a)}{n!}$$

that is, p is determined uniquely by (i), (ii), and (iii).

(b) $p(x) = b_0 + b_1(x - a) + b_2(x - a)^2$

where, as above,

$$b_0 = f(a), \quad b_1 = f'(a), \quad b_2 = \frac{f''(a)}{2!}$$

Since $f : x \to \sin x$, we have

$$f' : x \to \cos x \quad \text{and} \quad f'' : x \to -\sin x$$

so that

$$b_0 = \sin a, \quad b_1 = \cos a, \quad b_2 = -\frac{\sin a}{2!}$$

(c) The coefficients of p, expressed in powers of $x - a$ are given in part (a).

(d) If $g : x \to \cos x$, then arguing as above, for $n = 2$

$$p(x) = g(a) + g'(a)(x - a) + \frac{g''(a)}{2!}(x - a)^2$$

$$= \cos a - (\sin a)(x - a) - \left(\frac{\cos a}{2!}\right)(x - a)^2.$$

Teacher's Commentary

Chapter 5.

POWER, EXPONENTIAL AND LOGARITHMIC FUNCTIONS

In many calculus books the integral theorems are used to frame precise definitions of the power, exponential, and logarithmic functions. We begin by proceeding intuitively from the familiar algebraic properties, which have yet to be proved conclusively in all generality. Starting with the intuitive concept of power, we turn to the functions based upon power: exponential, logarithmic.

The main purpose of this chapter is to study the properties of the exponential functions $f : x \to a^x (a > 0)$ and their inverses, the logarithmic functions.

It is assumed that the student is familiar with the laws of exponents, in particular with

$$a^{r+s} = a^r a^s$$

and

$$(a^r)^s = a^{rs},$$

where r and s are rational numbers. Nevertheless, these matters are reviewed and used in connection with a concrete problem -- the growth of a colony of bacteria. In Section 5-4, a^x is given a meaning when x is irrational.

An alternate approach to exponential functions would have been to introduce and solve the functional equation $f(x + y) = f(x)f(y)$. We believe that this development could be very illuminating in the next course.

It is shown (in Section 5-5) that we can write an arbitrary number a as a power of 2 and that it is therefore sufficient to treat the single exponential function $x \to 2^x$. We begin with base 2 for simple concreteness. Later we shall expect the student to discover that exponential functions with bases other than e are not often used, since any exponential function is easily given by $x \to a^x = e^{\alpha x}$, where $\alpha = \log_e a$.

Solutions Exercises 5-1

1. (a) $x^5 \cdot x^{-2} = x^3$

   (b) $10^9 \cdot 10^{-7} = 10^2$

   (c) $a^{3/5} \cdot a^{5/3} = a^{34/15}$

   (d) $2^2(\frac{2^4}{2^8}) = 2^2(\frac{1}{2^4}) = (\frac{1}{2})^2$

   (e) $2^{3/5} \cdot 8^{4/3} = 2^{3/5} \cdot (2^3)^{4/3} = 2^{3/4+4} = 2^{19/4}$

   (f) $(a^{3/5})^{5/3} = a^1 = a$

   (g) $\frac{64^{2/3}}{32^{3/10}} = \frac{(2^6)^{2/3}}{(2^5)^{3/10}} = \frac{2^4}{2^{3/2}} = 2^{5/2}$

   (h) $\frac{3 \cdot 2^{1/2} - 8^{1/2}}{32^{1/2}} = \frac{2^{1/2}(3 - 4^{1/2})}{2^{1/2}16^{1/2}} = 2^0 \cdot \frac{1}{4} = (\frac{1}{2})^2$

   or $\frac{3 \cdot 2^{1/2} - 8^{1/2}}{32^{1/2}} = \frac{3 \cdot 2^{1/2} - 2 \cdot 2^{1/2}}{2^2 \cdot 2^{1/2}} = \frac{2^{1/2}(3 - 2)}{2^{1/2} \cdot 2^2} = (\frac{1}{2})^2$

   (i) $\frac{3^2 \cdot 2^2}{(6^{2/3})^0} = \frac{(3 \cdot 2)^2}{1} = (6)^2$

   (j) $\frac{5^{1.1}}{25^{.55}}(\frac{1}{125})^{-2/3} = \frac{5^{1.1}}{5^{1.10}}(\frac{1}{5^3})^{-2/3} = 5^0 \cdot \frac{1}{5^{-2}} = 5^2$

2. (a) If $8^m = (2^3)^2$, then $8^m = 8^2$ and $m = 2$.

   (b) If $8^m = 2^{(3^2)}$, then $2^{3m} = 2^9$ and $m = 3$.

   (c) If $2^{(4^5)} = 16^m$, then $2^{(4^5)} = (2^4)^m = 2^{4m}$ and $4^5 = 4m$ so that $m = 4^4 = 256$.

   (d) If $(2^4)^5 = 16^m$, then $2^{20} = 2^{4m}$ and $20 = 4m$, so that $m = 5$.

   (e) If $4^2 = 64^{2/3}$, then $4^{m-1} = (4^3)^{2/3}$ and $4^{m-1} = 4^2$.

   (f) If $5^{m-1} = 0.2$, then $5^{m-1} = 5^{-1}$ and $m = 0$.

   (g) If $(\frac{3}{2})^m = \frac{2}{3}$, then $(\frac{3}{2})^m = (\frac{3}{2})^{-1}$ and $m = -1$.

   (h) If $17^m = 1$, then $17^m = 17^0$ and $m = 0$.

3. $1000(8^{-2/3}) = 1000 \cdot \frac{1}{8^{2/3}} = 1000 \cdot \frac{1}{2^2} = 250$

$3(\frac{9}{4})^{-3/2} = 3(\frac{3}{2})^{-3} = 3(\frac{2}{3})^3 = 3 \cdot \frac{8}{27} = \frac{8}{9}$

4. $(4^{5/2})(8^{-1}) = (2^2)^{5/2}(2^3)^{-1} = 2^5 \cdot 2^{-3} = 2^2$

$(\frac{1}{2})^{-4/3} = (2^{-1})^{-4/3} = 2^{4/3}$

$(2^{-2/9})^9 = 2^{-2}$.

Since $x \to 2^x$ is an increasing function as x increases, in order of decreasing value from the left we have

$$2^2,\ 2^{4/3},\ 2^{2/3},\ 2^{-2},\ 2^{-3} \quad \text{or}$$

$$(4^{5/2})(8^{-1}),\ (\tfrac{1}{2})^{-4/3},\ 2^{2/3},\ (2^{-2/9})^9,\ 2^{-3}.$$

5. $2^{2.7} = 2^2 \cdot 2^{.7} = 4 \cdot 2^{7/10} = 4 \cdot \sqrt[10]{2^7} = 4 \cdot \sqrt[10]{128}$

6. (a) An increase in the exponent of 3 by $\frac{1}{4}$ corresponds to multiplication by r, say. If we start with the exponent 0 we have $3^0 = 1$. Then $3^{1/4} = r$, $3^{2/4} = r^2$, $3^{3/4} = r^3$, and $3^{4/4} = r^4$. Hence $r^4 = 3$ and $r = \sqrt[4]{3}$.

(b) A decrease in the exponent of 4 by 1 corresponds to division by 4. Since $4^1 = 4$, $4^0 = 1$, $4^{-1} = \frac{1}{4}$, $4^{-2} = \frac{1}{16}$, $4^{-3} = \frac{1}{64}$.

(c) A decrease in the exponent of 2 by $\frac{1}{2}$ corresponds to a division by $2^{1/2}$. Since $2^0 = 1$,

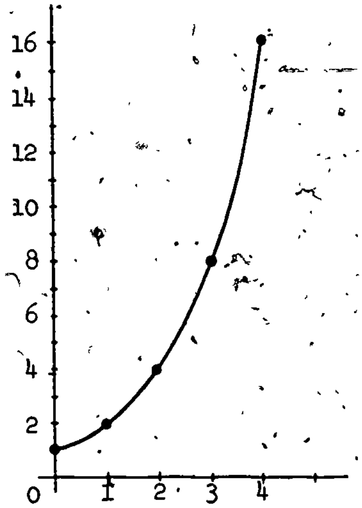
$$2^{0-\frac{1}{2}} = \frac{1}{2^{1/2}}.$$

Solutions Exercises 5-2

1. Let $N(n) = 10^6(2^n)$

n	$N(n)$ in millions
0	1
1	2
2	4
3	8
4	16

$N(n)$ in millions



2. $N(n) = 10^6(2^n)$ so that

$$\frac{N(n+5)}{N(n+2)} = \frac{10^6(2^{n+5})}{10^6(2^{n+2})} = 2^3 = 8$$

3. $$\frac{N(n+7)}{N(n-3)} = \frac{10^6(2^{n+7})}{10^6(2^{n-3})} = 2^{10} = 1024$$

4. If $k = N(n + 100) = 10^6(2^{100})$, then $\frac{k}{4} = \frac{10^6(2^{100})}{2^2} = 10^6(2^{98})$.

Thus, after 98 days there were $\frac{k}{4}$ present.

5. $N(n) = N_0 \cdot a^n$

$200{,}000 = N(n+3) = N_0(a^{n+3})$

and

$1{,}600{,}000 = N(n + \frac{9}{2}) = N_0\, a^{n+\frac{9}{2}}$

Thus

$$\frac{N(n+\frac{9}{2})}{N(n+3)} = a^{3/2} = 8, \text{ and } a = 4; \text{ hence}$$

$N(n) = N_0(4^n)$, $N(3) = N_0(4^3) = 200{,}000$, and $N_0 = 200{,}000(4^{-3})$.

The formula for $N(n)$ becomes

$$N(n) = 200{,}000 \cdot 4^{n-3}.$$

(a) If $n = 5$, $N(5) = 200,000 \cdot 4^{5-3} = 3,200,000$

(b) If $n = \frac{3}{2}$, $N(\frac{3}{2}) = 200,000 \cdot 4^{(3/2)-3}$

$$= 200,000 \cdot 4^{-3/2} = 200,000 \cdot \frac{1}{8}$$

$$= 25,000.$$

(c) If $N(n) = 800,000$, then $200,000 \cdot 4^{n-3} = 800,000$ and $4^{n-3} = 4^1$. Therefore, $n - 3 = 1$, and $n = 4$.

6. $N(t) = 10^5 \cdot 2^t$

(a) $N(2) = 10^5 \cdot 2^2 = 4 \cdot 10^5$

$N(4) = 10^5 \cdot 2^4 = 16 \cdot 10^5$

(b) $N(-1) = 10^5 \cdot 2^{-1} = \frac{1}{2} \cdot 10^5$

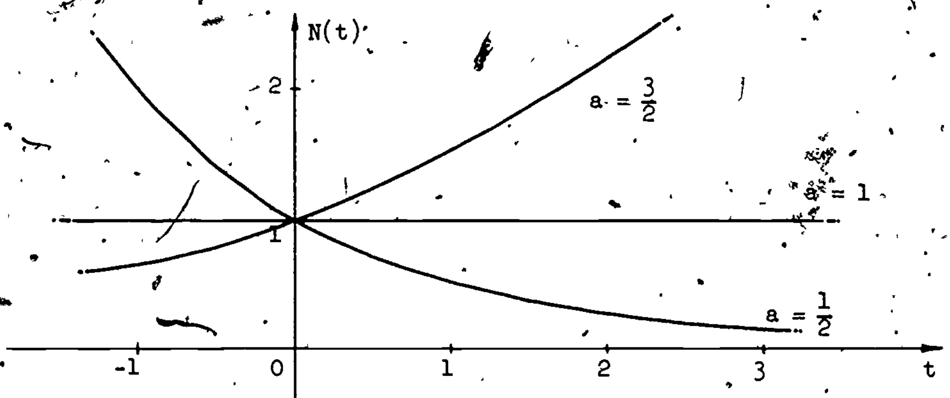
$N(-2) = 10^5 \cdot 2^{-2} = \frac{1}{4} \cdot 10^5$

(c) $N(-\frac{1}{2}) = 10^5 \cdot 2^{-1/2} = \frac{1}{\sqrt{2}} \cdot 10^5$

$N(\frac{1}{2}) = 10^5 \cdot 2^{1/2} = \sqrt{2} \cdot 10^5$

$$\frac{N(\frac{1}{2})}{N(-\frac{1}{2})} = 2$$

7. (a) If $N(t) = N_0 a^t$ is to represent growth, a must exceed one.



(b) $\frac{N(t+1)}{N(t)} = \frac{N_0 a^{t+1}}{N_0 a^t} = \frac{N_0 a^t \cdot a}{N_0 a^t} = \frac{N(t) \cdot a}{N(t)} = a$

$\frac{N(t+2)}{N(t+1)} = \frac{N_0 a^{t+2}}{N_0 a^{t+1}} = \frac{N_0 a^{t+1} \cdot a}{N_0 a^{t+1}} = \frac{N(t+1) \cdot a}{N(t+1)} = a$

$\frac{N(t+3)}{N(t+2)} = \frac{N_0 a^{t+3}}{N_0 a^{t+2}} = \frac{N_0 a^{t+2} \cdot a}{N_0 a^{t+2}} = \frac{N(t+2) \cdot a}{N(t+2)} = a$

(c) $\frac{N(t+n+1)}{N(t+n)} = \frac{N_0 a^{t+n} \cdot a}{N(t+n)} = \frac{N(t+n) \cdot a}{N(t+n)} = a$ i.e., Rate of growth is proportional to amount present.

(d) $N(1) = N_0 a^1 = 10^5$

$N(2) = N_0 a^2 = 10^6$

$\frac{N(2)}{N(1)} = \underline{\underline{a = 10}}$

Therefore:

$N(1) = N_0 10^1 = 10^5$

$N(2) = N_0 10^2 = 10^6$ $\underline{\underline{N_0 = 10^4}}$

8. $N(t) = N_0 a^t$

(a) At time $t = 0$, $n(0) = N_0 a^0 = N_0$.

(b) For decay, $\underline{a}$ must be less than 1. See Number 7 and its solution.

(c) $\frac{N(t+1)}{N(t)} = \frac{N_0 a^{t+1}}{N_0 a^t} = a < 1$

(d) If $N(t) = \frac{N_0}{2} = N_0 (\frac{1}{2})^t$

then $\frac{1}{2} = (\frac{1}{2})^t$ and $t = 1$.

Solutions Exercises 5-3

1. (a) $2^{5/4} = 2^{1.25} = 2 \cdot 2^{.25} \approx 2(1.189) = 2.378$

   (b) $2^{5/4} = 2\sqrt{\sqrt{2}} \approx 2\sqrt{1.414} \approx 2(1.189) = 2.378$

2. (a) $2^{1.15} = 2 \cdot 2^{.15} \approx 2(1.110) = 2.220$

   (b) $2^{2.65} = 2^{2} \cdot 2^{.65} \approx 4(1.569) = 6.276$

   (c) $2^{0.58} = 2^{.55} \cdot 2^{.03} \approx (1.464)(1.021) \approx 1.495$

   (d) $2^{-0.72} = 2^{-1+.28} = \frac{1}{2}(2^{.25})(2^{.03})$

   $\approx .5(1.189)(1.021) \approx .607$

3. (a) $8^{.84} = (2^{3})^{.84} = 2^{2.52} = 2^{2} \cdot 2^{.5} \cdot 2^{.02}$

   $\approx 4(1.414)(1.014) \approx 5.735$

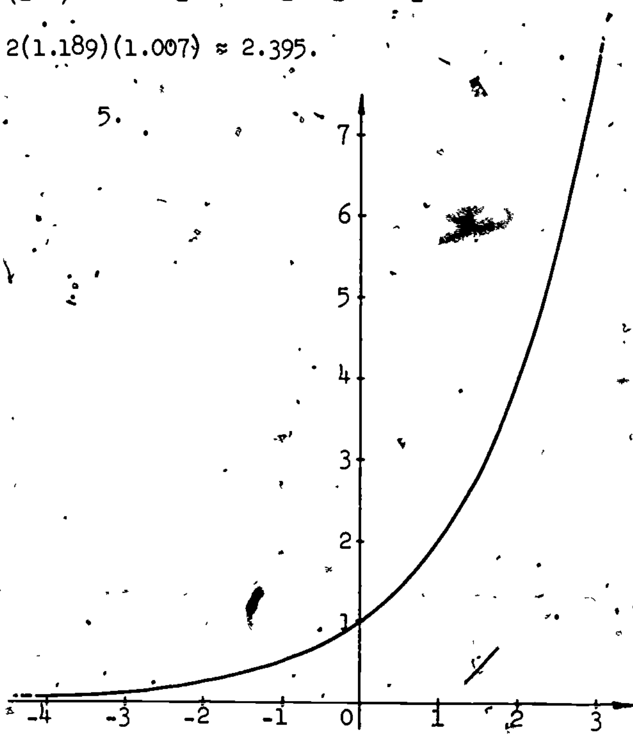
   (b) $0.25^{-0.63} = (2^{-2})^{-0.63} = 2^{1.26} = 2^{1} \cdot 2^{.25} \cdot 2^{.01}$

   $\approx 2(1.189)(1.007) \approx 2.395.$

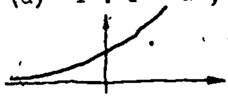
4.

r	2^r
-4.0	.0625
-3.6	.0825
-3.2	.109
-2.8	.144
-2.4	.189
-2.0	.250
-1.6	.330
-1.2	.435
1.4	2.639
1.8	3.482
2.2	4.595
2.6	6.063
3.0	8.000

5.



6. (a) $f : r \to a^r$, $a > 0$, ($\underline{a}$ constant)

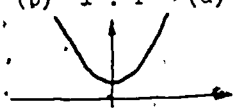


If $a > 1$, f is increasing

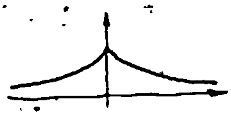
If $0 < a < 1$, f is decreasing

If $a = 1$, f is constant.

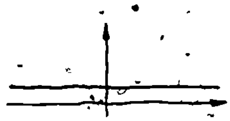
(b) $f : r \to (a)^{r^2}$, $a > 0$, ($\underline{a}$ constant)



If $a > 1$, f is decreasing for $r < 0$ and increasing for $r > 0$.

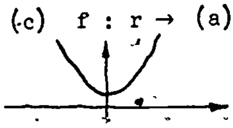


If $0 < a < 1$, f is increasing for $r < 0$ and f is decreasing for $r > 0$.



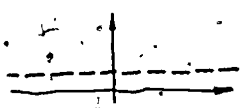
If $a = 1$, f is constant.

(c) $f : r \to (a)^{-r^2}$, $a > 0$, ($\underline{a}$ constant)



If $0 < a < 1$, f is decreasing for $r < 0$ and f is increasing for $r > 0$.

If $a > 1$, f is increasing for $r < 0$, and f is decreasing for $r > 0$.



If $a = 1$, f is constant.

(d) $f : r \to (2b + 3)^r$, ($\underline{b}$ constant, $2b + 3 > 0$) i.e., $b > -\frac{3}{2}$

Increasing: $2b + 3 > 1$, $\underline{b > -1}$

Decreasing: $2b + 3 < 1$, $\underline{-\frac{3}{2} < b < -1}$

Constant: $2b + 3 = 1$; $\underline{b = -1}$

7. (a) $2^n = (1+1)^n = 1^n + \frac{n}{1}(1)^{n-1}(1) + \frac{n(n-1)}{1 \cdot 2}(1)^{n-2}(1)^2 + \ldots$

Since all terms are positive, and if $n \geq 2$ there will be at least 3 terms on the right hand side, we may deduce the inequality

$$2^n > \frac{n(n-1)}{2}, \qquad n \geq 2$$

(b) (i) From (a) $2^n > \frac{n(n-1)}{2}$, $n \geq 2$

Let $n = 100$; $\therefore$ $2^{100} > \frac{100 \cdot 99}{2}$

and $\frac{2^{100}}{100} > \frac{99}{2}$

(ii) In (a) let $n = 10{,}000$; $\therefore$ $2^{10{,}000} > \frac{10{,}000(9{,}999)}{2}$

and $\frac{2^{10{,}000}}{10{,}000} > \frac{9{,}999}{2}$

(c) From (a) we have

$$2^n > \frac{n(n-1)}{2}$$

If $n > 0$, then $\frac{2^n}{n} > \frac{n-1}{2}$

and $\frac{2^n}{n} > \frac{n}{2} - \frac{1}{2}$

As n increases, the contribution of $(-\frac{1}{2})$ is negligible, and $\frac{n}{2}$ increases without bound. Therefore, since $\frac{2^n}{n}$ is greater than $\frac{n}{2} - \frac{1}{2}$, $\frac{2^n}{n}$ will also increase without bound.

21\. (a) In (4) $\log_a b = \frac{\log_c b}{\log_c a}$ substitute x for b, and b for c.

We then have $\log_a x = \frac{\log_b x}{\log_b a}$

$\therefore \log_b x = (\log_b a)\log_a x$, or using (5)

$\log_a x = (\log_a b)\log_b x$.

i.e., In the first relationship $\log_b a$ is the constant of proportionality; and in the second, $\log_a b$.

(b) $\log_a x = 0$, if $x = 1$. Therefore the x-intercept is (1,0).

22\. $\log_a b = \frac{\log_c b}{\log_c a}$ (4). Let $c = b$, and we have $\log_a b = \frac{\log_b b}{\log_b a}$.

$\therefore \log_a b = \frac{1}{\log_b a}$ (5).

23\. (a) $x^{3/2} = a^{3/2 \log_a x}$ (6).

With $a = 2$, we have $x^{3/2} = 2^{3/2 \log_2 x}$

With $a = e$, we have $x^{3/2} = e^{3/2 \log_e x}$

(b) $(x^x)^x = (x)^{x^2} = a^{x^2 \log_a x}$

With $a = 10$, we have $(x^x)^x = 10^{x^2 \log_{10} x}$

(c) $x^{2x} = a^{2x \log_a x}$

With $a = e$, we have $x^{2x} = e^{2x \log_e x}$

(d) $x^{(x^x)} = a^{(x^x)\log_a x}$. But $x^x = a^{x \log_a x}$

$\therefore x^{(x^x)} = a^{(a^{(x \log_a x)})\log_a x}$ With $a = 2$, we have

$x^{(x^x)} = 2^{[(\log_2 x) \cdot 2^{x \log_2 x}]}$

Solutions Exercises 5-4

1. Reading from the graph, we find

$$2^{1.15} \approx 2.2; \quad 2^{2.65} \approx 6.3^{-}; \quad 2^{0.58} \approx 1.5; \quad 2^{-0.72} \approx 0.6.$$

2. The values obtained in Exercises 5-3, Number 2 were 2.220, 6.276, 1.495, and 0.607, respectively. A comparison should reveal a difference of not more than 0.2.

3. (a) $2^{\sqrt{3}} \approx 2^{1.73} \approx 3.3$

   (b) $2^{\pi} \approx 2^{3.14} \approx 8.8$

   (c) $2^{-\pi/4} \approx 2^{-.79} \approx .6$

4. No, there is no real number x such that $2^x = 0$. We give two arguments to support this statement. The second is very neat, but more sophisticated.

   (a) There is no such real number $x \geq 0$ since $f : x \to 2^x$ is an increasing function and $2^0 = 1$ so that $f(x) \geq 1$ for $x \geq 0$. If $x < 0$ we set $t = -x$; hence, $2^x = \frac{1}{2^t}$. Since there is no number 2^t such that $\frac{1}{2^t} = 0$, there is no real number x such that $2^x = 0$.

   (b) If x is a real number such that $2^x = 0$, and if y is any real number whatever, then $y - x$ is a real number, 2^{y-x} is therefore defined, and we have

$$2^{y-x} \cdot 2^x = 2^{y-x} \cdot 0 = 0,$$

   but

$$2^{y-x} \cdot 2^x = 2^y,$$

   therefore

$$2^y = 0$$

   for every real number y. But $2^1 = 2$, so this is impossible.

5. (a) If $2^x = 6$, $x \approx 2.6$

   (b) If $2^x = .4$, $x \approx -1.3$

   (c) If $2^x = 3.8$, $x \approx 1.9$

   (d) If $2^x = 3$, $x \approx 1.6$

   (e) If $2^x = 2.7$, $x \approx 1.4$

Solutions Exercises 5-5

1. We first write $3.4 = 2(1.7)$. In order to express 1.7 as a power of 2 we may,

   (a) use the graph and read $2^{.77} \approx 1.7$,

   (b) interpolate in Table 4-2 between the entries 1.68179 and 1.74110, obtaining $x \approx .77$;

   (c) if we do not object to the calculation involved, divide 1.7 by 1.682 (since entry $1.68179 \approx 1.682$) obtaining $1.7 \approx (1.682)(1.0107)$.

   Using the table again we note that

$$2^{.01} < 1.0107 < 2^{.02};$$

   hence $$1.7 \approx (2^{.75})(2^{.02}) = 2^{.77}.$$

   Getting back to the given exercise we may now express $3.4 = 2(1.7)$ as $2^1(2^{.77}) = 2^{1.77}$ (approximately). We should note that the graph gives a satisfactory approximation.

2. We write $2.64 = 2^1(1.32)$; then, since

$$2^{.40} \approx 1.32,$$

$$2.64 \approx 2^1(2^{.40}) = 2^{1.40}.$$

   Thus, $(2.64)^{0.3} \approx (2^{1.40})^{0.3} = 2^{0.42} = (2^{0.40})(2^{0.02})$.

   From Table 5-3, we have

$$2^{0.4} \approx 1.320, \quad 2^{0.02} \approx 1.014;$$

   hence $(2.64)^{0.3} \approx (2^{0.4})(2^{0.02}) \approx (1.32)(1.014) \approx 1.34$.

   Alternatively, the graph of $x \to 2^x$ may be used twice: first to see that $2.64 \approx 2^{1.4}$; then to find

$$(2^{1.4})^{0.3} = 2^{0.42}.$$

   Using the graph, we obtain the result

$$2^{0.42} \approx 1.3.$$

3. We first write $6.276 = 4(1.569)$. The table or graph gives

$$1.569 \approx 2^{0.65}$$

so that

$$6.276 = 2^2(1.569) \approx 2^2(2^{0.65}) = 2^{2.65}.$$

Then

$$\begin{aligned}(6.276)^{-0.6} \approx (2^{2.65})^{-0.6} &= 2^{-1.59}\\ &= 2^{-2+.41} = 2^{-2}(2^{.41})\\ &\approx \tfrac{1}{4}(1.33) \approx 0.33.\end{aligned}$$

Alternatively, to use the graph we round

$$6.276 \approx 6.28 \quad \text{and read off} \quad 2^{2.65} \approx 6.28.$$

Then since

$$(6.28)^{-0.6} \approx (2^{2.65})^{-0.6} = 2^{-1.59} \approx 2^{-1.6},$$

we refer to the graph again and read off

$$2^{-1.6} \approx 0.35.$$

The result is satisfactory, and quickly obtained.

4. Since $5.2 = 4(1.3)$ and $1.3 \approx 2^{.38}$

$$5.2 \approx 2^2(2^{.38}) = 2^{2.38}$$

Thus,

$$(5.2)^{2.6} \approx (2^{2.38})^{2.6} = 2^{6.188} \approx 2^6(2^{0.19}).$$

But

$$2^{0.19} \approx 1.14,$$

hence

$$2^6(2^{0.19}) \approx 2^6(1.14) = 72.96.$$

We have

$$(5.2)^{2.6} \approx 73.$$

5. Prove that if $0 < a < 1$ and $v > u$, then $a^v < a^u$.

Proof: Let $b = \frac{1}{a}$. Then $b > 1$ and $b^v > b^u$, if $v > u$.

Dividing by $b^v b^u$

$$\frac{1}{b^u} > \frac{1}{b^v}$$

$$\left(\frac{1}{b}\right)^u > \left(\frac{1}{b}\right)^v$$

and finally $\quad a^u > a^v$.

6. (a) $x \to 4^x$; $4 = 2^2$; $\therefore \alpha = 2$

$\therefore$ $\underline{x \to 2^{2x}}$

(b) $x \to (3.60)^x = 2^{\alpha x}$

$\therefore 2^\alpha = 3.60 = 2(1.80) \doteq 2^1(2^{0.85}) = 2^{1.85}$

i.e., $\alpha = 1.85$, and the function $x \to (3.60)^x$, may be expressed as $\underline{x \to 2^{1.85x}}$.

(c) $x \to (5.736)^x = (2^\alpha)^x$

$$\begin{aligned}\therefore 2^\alpha &= 5.736\\ &= 4(1.434)\\ &= 2^2 \cdot 2^{0.52}\\ &= 2^{2.52}\end{aligned}$$

Linear interpolation:

Table value: $2^{0.50} = 1.414$

$2^{0.52} = 1.434 \quad \frac{20}{50}(.05) = .02$

Table value: $2^{0.55} = 1.464$

i.e., $\alpha = 2.52$, and the function $x \to (5.736)^x$ may be expressed as $\underline{x \to 2^{2.52x}}$.

(d) $x \to (0.420)^x = (2^\alpha)^x$

$$\begin{aligned}\therefore 2^\alpha &= .420\\ &= (.5)(.840)\\ &\doteq (2^{-1})(2^{-0.25})\\ &\doteq 2^{-1.25}\end{aligned}$$

i.e., $\alpha = -1.25$, and the function $x \to (0.420)^x$ may be expressed as $x \to 2^{-1.25x}$.

7. The function $x \to a^x$ may be written as $x \to 2^{\alpha x}$.
The function $x \to b^x$ may be written as $x \to 2^{\beta x}$

(a) If $a < b$, then $2^\alpha < 2^\beta$. Therefore, since 2^x is an increasing function, it follows that $\underline{\alpha < \beta}$.

(b) As $|b - a|$ decreases $|\beta - \alpha|$ also decreases.

	For these values of b			
	4	3	2.3	2.04
$\beta =$	2	1.6	1.2	1.03

	a
	2
$\alpha =$	1

If $\underline{b = 4}$, then $2^\beta = 4$ and $\underline{\beta = 2}$

If $\underline{b = 3}$, then $2^\beta = 3 = 2(1.5) \approx (2^1)(2^{0.6}) = 2^{1.6}$,

and $\underline{\beta \approx 1.6}$

If $\underline{b = 2.3}$, then $2^\beta = 2.3 = 2(1.15) \approx (2^1)(2^{0.2}) = 2^{1.2}$,

and $\underline{\beta \approx 1.2}$

If $\underline{b = 2.04}$, then $2^\beta = 2.04 = 2(1.02) \approx (2^1)(2^{0.03}) = 2^{1.03}$

and $\underline{\beta \approx 1.03}$

8. If the solution of Number 6 above is used, this question has been answered.

(a) Positive, because if $x \to a$ is increasing, then $a > 1$, and $\alpha > 0$.

(b) Negative, because if $x \to a^x$ is decreasing, then $0 < a < 1$ and $\alpha < 0$.

(c) $\frac{f(x + 1)}{f(x)} = \frac{a^{x+1}}{a^x} = \underline{a = 2^\alpha}$ (independent of x)

If $\alpha > 0$, then $\frac{f(x + 1)}{f(x)} > 1$

If $\alpha < 0$, then $0 < \frac{f(x + 1)}{f(x)} < 1$

(d) $a^x = 2^{\alpha x} = (2^x)^\alpha > (2^x)^2$ since $\alpha > 2$

$\therefore$ If $\alpha > 2$, $a^x > (2^x)^2$

9. (a) The graph of $x \to a^x$ $(a > 0)$ crosses the y-axis at the point $(0,1)$. This y-intercept is independent of a.

(b) If $x \to a^x$ and $x \to 2(2a)^x$, then $a^x = 2 \cdot 2^x \cdot a^x$.
It follows that $1 = 2 \cdot 2^x$ and $2^{-1} = 2^x$. $\therefore$ $\underline{x = -1}$

(c) If $x \to a^x$ and $x \to b(ba)^x$, $(b(Re) > 0)$
then $a^x = b \cdot b^x \cdot a^x$.
It follows that $1 = b \cdot b^x$ and $b^{-1} = b^x$. $\therefore$ $\underline{x = -1}$

(d) If $x \to a^x$ and $x \to b^n(ba)^x$, $(b(Re) > 0,$ $n(Re) > 0)$
then $a^x = b^n \cdot b^x \cdot a^x$.
It follows that $1 = b^n \cdot b^x$ and $b^{-n} = b^x$. $\therefore$ $\underline{x = -n}$

Solutions Exercises 5-6

1. $\log_2 (x \cdot \frac{1}{x}) = \log_2 1 = 0$. For any real number $x > 0$,

$\log_2 (x \cdot \frac{1}{x}) = \log_2 x + \log_2 (\frac{1}{x})$

$\therefore \log_2 x + \log_2(\frac{1}{x}) = 0$. It follows that

$\log_2 (\frac{1}{x}) = -\log_2 x$ for $x > 0$.

2. For any real numbers $x_1 > 0$, $x_2 > 0$,

$$\log_2 \frac{x_1}{x_2} = \log_2 (x_1)(\frac{1}{x_2}) = \log_2 x_1 + \log_2 (\frac{1}{x_2})$$

$$= \log_2 x_1 - \log_2 x_2, \text{ from Exercise 1.}$$

3. Let $a = \log_2 x^y$ and $b = \log_2 x$

$\therefore 2^a = x^y$ and $2^b = x$. But $2^{by} = x^y$

$\therefore 2^a = 2^{by}$ and $a = by$

Substituting for a and b it follows that

$\log_2 x^y = y \log_2 x$

4. (a) If $\log_2 x > 0$, then $x > 1$; if $\log_2 x < 0$, then $0 < x < 1$.

(b) (i) If $\log_2 2x > 0$, then $x > \frac{1}{2}$;

if $\log_2 2x < 0$, then $0 < x < \frac{1}{2}$.

(ii) If $\log_2 (-x) > 0$, then $x < -1$

if $\log_2 (-x) < 0$, then $-1 < x < 0$.

(iii) If $\log_2 (x - 1) > 0$, then $x > 2$

if $\log_2 (x - 1) < 0$, then $1 < x < 2$.

(iv) If $\log_2 (1 - x) > 0$, then $x < 0$

if $\log_2 (1 - x) < 0$, then $0 < x < 1$.

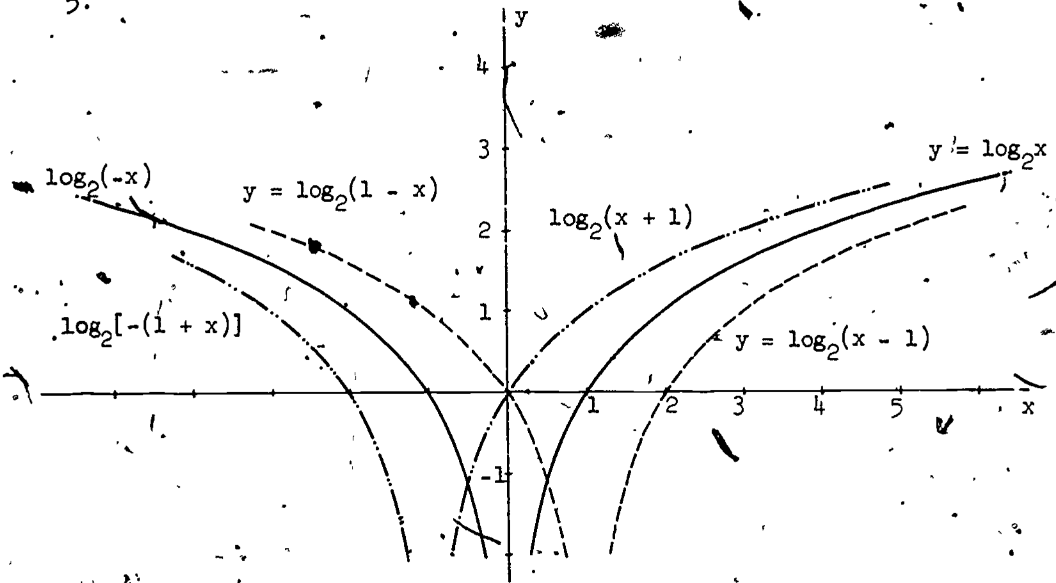
(v) If $\log_2(2x - 3) > 0$, then $x > 2$

if $\log_2(2x - 3) < 0$, then $\frac{3}{2} < x < 2$

(vi) If $\log_2(3 - 2x) > 0$, then $x < 1$

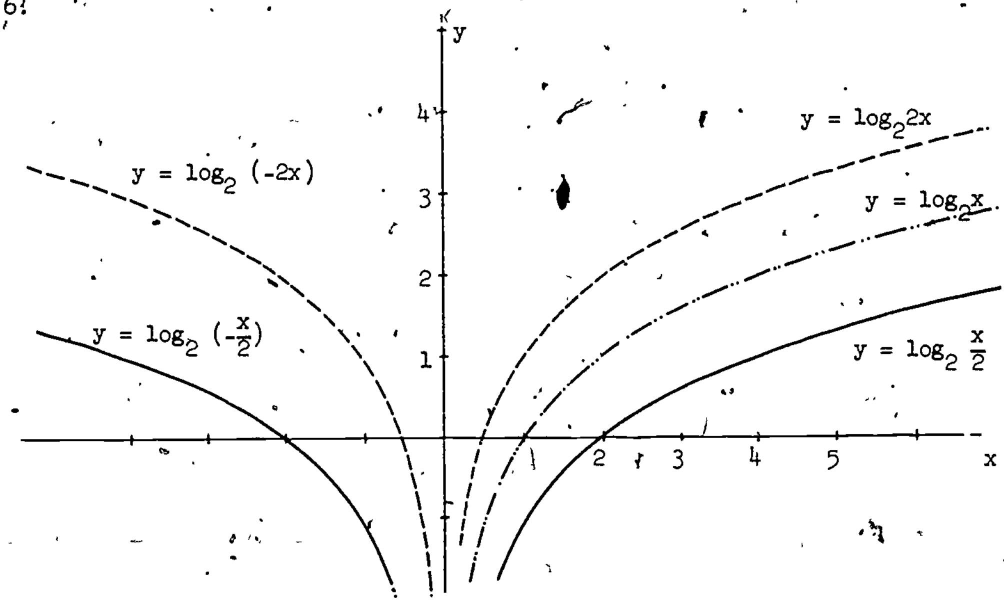
if $\log_2(3 - 2x) < 0$, then $1 < x < \frac{3}{2}$.

5.



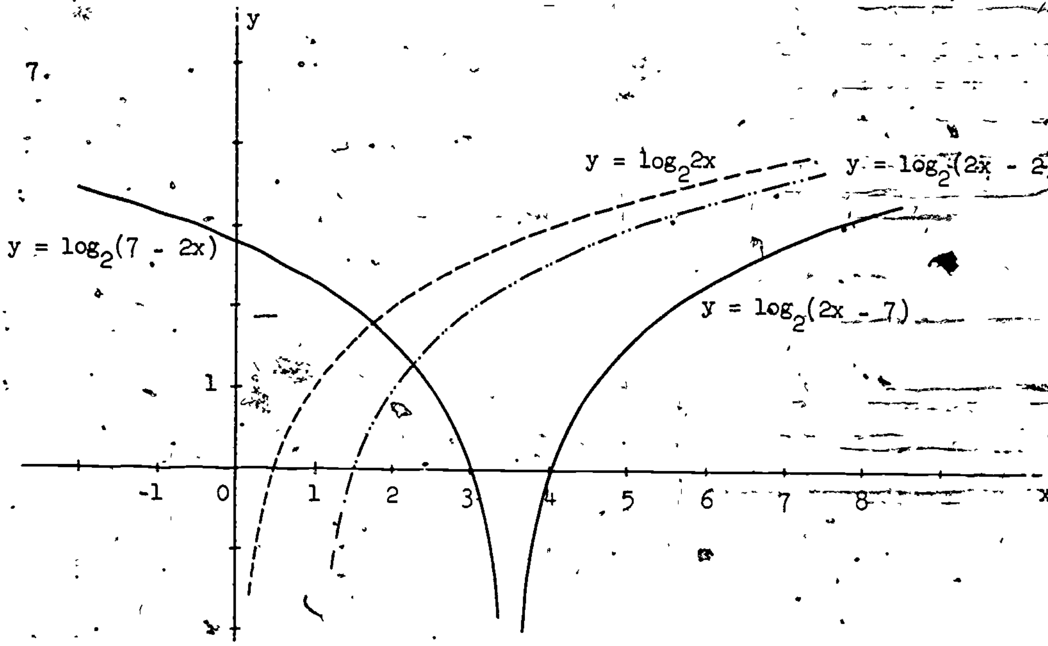
y
4
3
2
1
-1
1
2
3
4
5
x
y = log2x
log2(-x)
y = log2(1 - x)
log2(x + 1)
log2[-(1 + x)]
y = log2(x - 1)

6.



y
4
3
2
1
2
3
4
5
x
y = log2 2x
y = log2 (-2x)
y = log2x
y = log2 (-x/2)
y = log2 x/2

7.



8. (a) $2^b = a$

(b) $\log_2 x^3 = 5k$

$\therefore\ 2^{5k} = x^3$

(c) $\log_2 zw = 4$

$\therefore\ 2^4 = zw$

(d) $\log_2 3^{-2} = -t$

$\therefore\ 2^{-t} = 3^{-2}$

or $2^t = 9$

9. (a) $\log_2 13 = x$

(b) $(2)^{2k} = \frac{3}{5} = 0.6$

$\therefore\ \log_2 (0.6) = 2k$

(c) $(2^{-1})^{-3} = mn$

$(2)^{+3} = mn$

$\therefore\ \log_2 mn = 3$

(d) $(2)^{3/2} = x^y$

$\therefore\ \log_2 x^y = \frac{3}{2}$

or $y \log_2 x = \frac{3}{2}$

10. (a) $\log_2 2 = 1$

(b) $\log_2 4 = 2$

(c) $\log_2 8 = 3$

(d) $\log_2 16 = 4$

(e) $\log_2 1 = 0$

(f) $\log_2 \frac{1}{2} = -1$

(g) $\log_2 \frac{1}{4} = -2$

(h) $\log_2 \frac{1}{8} = -3$

11. (a) $1 < \log_2 3 < 2$, since $2^1 < 3 < 2^2$.
   (b) $2 < \log_2 5 < 3$, since $2^2 < 5 < 2^3$.
   (c) $2 < \log_2 6 < 3$, since $2^2 < 6 < 2^3$.
   (d) $2 < \log_2 7 < 3$, since $2^2 < 7 < 2^3$.
   (e) $3 < \log_2 9 < 4$. since $2^3 < 9 < 2^4$.
   (f) $3 < \log_2 10 < 4$, since $2^3 < 10 < 2^4$.
   (g) $3 < \log_2 13 < 4$. since $2^3 < 13 < 2^4$.
   (h) $4 < \log_2 18 < 5$, since $2^4 < 18 < 2^5$.
   (i) $4 < \log_2 31 < 5$, since $2^4 < 31 < 2^5$.
   (j) $5 < \log_2 34 < 6$, since $2^5 < 34 < 2^6$.
   (k) $5 < \log_2 60 < 6$, since $2^5 < 60 < 2^6$.
   (ℓ) $6 < \log_2 99 < 7$, since $2^6 < 99 < 2^7$.
   (m) $-1 < \log_2 (0.9) < 0$, since $2^{-1} < (0.9) < 2^0$.
   (n) $-1 < \log_2 (\frac{3}{4}) < 0$, since $2^{-1} < (\frac{3}{4}) < 2^0$.
   (o) $-2 < \log_2 (\frac{3}{8}) < -1$, since $2^{-2} < (\frac{3}{8}) < 2^{-1}$.
   (p) $-3 < \log_2 (0.18) < -2$, since $2^{-3} < (0.18) < 2^{-2}$.

12. (a) From Number 4 (a): $1 < \log_2 3 < 2$.

   Using Table 5-3:

$$\begin{aligned}\log_2 3 &= \log_2 (2)(1.5)\\ &= \log_2 (2) + \log_2 (1.5)\\ &\approx 1 + 0.59\\ &\approx 1.59\end{aligned}$$

   Interpolation:

$$\left.\begin{aligned}\log_2 .55 &= 1.464\\ \log_2 .59 &= 1.500\end{aligned}\right\}36 \qquad \left.\log_2 .60 = 1.515\right\}51$$

$$\frac{36}{51} \times .05 \approx .035^+ \approx .04$$

   (b) From Number 4 (b): $2 < \log_2 5 < 3$.

   Using Table 5-3:

$$\begin{aligned}\log_2 5 &= \log_2 4(1.25)\\ &= \log_2 4 + \log_2 1.25\\ &\approx 2 + 0.33\\ &\approx 2.33\end{aligned}$$

(c) From Number 4 (g): $3 < \log_2 13 < 4$.

Using Table 5-3:

$$\begin{aligned} \log_2 13 &= \log_2 (8)(1.625) \\ &= \log_2 (8) + \log_2 (1.625) \\ &\approx 3 + .70 \\ &\approx 3.70 \end{aligned}$$

13. (a) $\log_2 6 = \log_2 2 \cdot 3 = \log_2 2 + \log_2 3 \approx 1 + 1.59 = 2.59$

(b) $\log_2 12 = \log_2 4 \cdot 3 = \log_2 4 + \log_2 3 \approx 2 + 1.59 = 3.59$

(c) $\log_2 24 = \log_2 8 \cdot 3 = \log_2 8 + \log_2 3 \approx 3 + 1.59 = 4.59$

(d) $\log_2 9 = \log_2 3^2 = 2 \log_2 3 \approx 2(1.59) = 3.18$

(e) $\log_2 27 = \log_2 3^2 = 3 \log_2 3 \approx 3(1.59) = 4.77$

(f) $\log_2 169 = \log_2 13^2 = 2 \log_2 13 \approx 2(3.70) = 7.40$

(g) $\log_2 54 = \log_2 2 . 3^3 = \log_2 2 + 3 \log_2 3 \approx 1 + 4.77 = 5.77$

(h) $\log_2 36 = \log_2 4 \cdot 9 = 2 \log_2 2 + 2 \log_2 3 \approx 2 + 3.18 = 5.18$

(i) $\log_2 52 = \log_2 4 \cdot 13 = 2 \log_2 2 + \log_2 13 \approx 2 + 3.70 = 5.70$

14. (a) $2^0 = x$

$\therefore\ x = 1$

(b) $2^4 = x$

$\therefore\ x = 16$

(c) $2^{-1} = x$

$\therefore\ x = \frac{1}{2}$

(d) $2^{1/2} = x$

$\therefore\ x = \sqrt{2}$

(e) $2^{-1/2} = x$

$\therefore\ x = \frac{1}{\sqrt{2}}$

(f) $2^{3/4} = x$

$\therefore\ x = \sqrt[4]{2^3}$

or $x = \sqrt[4]{8}$

15. Briefly, repeating earlier development

$$2^n = (1 + 1)^n = 1 + \frac{n}{1} + \frac{n(n - 1)}{1 \cdot 2} + \cdots$$

$$\therefore\ 2^n > \frac{n(n - 1)}{2}$$

$$\log_2 2^n > \log_2 \frac{n(n - 1)}{2}$$

$$n \log_2 2 > \log_2 n(n - 1) - \log_2 2$$

$$n > \log_2 n(n - 1) - 1$$

$$\therefore\ \log_2 n(n - 1) < n + 1$$

or alternately

if $2^n > \frac{n(n - 1)}{2}$

$$2^{n+1} > n(n - 1)$$

$$\therefore\ (n + 1)\log_2 2 > \log_2 n(n - 1)$$

and $\qquad n + 1 > \log_2 n(n - 1).$

Solutions Exercises 5-7

1. (a) $a^{\log_a 3} = 3$

(b) $a^{2 \log_a 3} = 3^2$

(c) $a^{1/2 \log_a 3} = \sqrt{3}$

(d) $8^{\log_2 5}$
$2^{3 \log_2 5} = 125$

(e) $16^{\log_4 2}$
$4^{2 \log_4 2} = 4$

(f) $32^{\log_2 \sqrt[5]{2}}$
$2^{5[\frac{1}{5} \log_2 2]} = 2$

2. $32 = 4^x \Longleftrightarrow x = \frac{5}{2}$

3. $a \cdot a^m = a^{1+m} = (a^2)^m = a^{2m} \Longleftrightarrow m = 1$

4. $\log_a (x \cdot \frac{1}{x}) = \log_a 1 = 0$. For any real number $x > 0$,

$\log_a (x \cdot \frac{1}{x}) = \log_a x + \log_a (\frac{1}{x})$; since $\log_a x + \log_a (\frac{1}{x}) = 0$,

$\log_a (\frac{1}{x}) = -\log_a x$ for $x > 0$.

5. For any real numbers $x_1 > 0$, $x_2 > 0$,

$$\log_a (\frac{x_1}{x_2}) = \log_a (x_1)(\frac{1}{x_2}) = \log_a x_1 + \log_a (\frac{1}{x_2})$$

$$= \log_a x_1 - \log_a x_2, \text{ from Exercise 4.}$$

6. If $f : x \to a^x$, $f(1) = a$, $f^{-1}(a) = 1$. In other words, $\log_a a = 1$.

7. (a) $10^y = 35$

(b) $2^x = 25$

(c) $\log_{10} 5^2 = x$, so that $10^x = 25$, or $10^{x/2} = 5$

8. $\log_{10} 5 = \log_{10} 10 - \log_{10} 2 \approx 1 - 0.3010 = 0.6990$

$\log_{10} (\frac{1}{2}) = \log_{10} 1 - \log_{10} 2 \approx 0 - 0.3010 = -0.3010$

$\log_{10} (\frac{25}{4}) = \log_{10} 5^2 - \log_{10} 2^2 = 2 \log_{10} 5 - 2 \log_{10} 2 \approx 0.7960$

$\log_{10} (\frac{128}{5}) = 7 \log_{10} 2 - \log_{10} 5 \approx 2.1070 - 0.6990 = 1.4080$

9. (a) $\log_{125} 5 = \frac{1}{3}$

(b) $\log_{10} 0.01 = -2$

(c) $\log_{27} 81 = \frac{4}{3}$

(d) $\log_{0.04} (0.008) = \frac{3}{2}$

(e) $\log_{16} 2 = \frac{1}{4}$ or $\log_{\sqrt{16}} 2 = \frac{1}{2}$

10. $\log_6 (x + 9)(x) = \log_6 36$, hence $x^2 + 9x = 36$

$x^2 + 9x - 36 = 0 \Leftrightarrow x = 3$ or $x = -12$.

The only root of the given equation is 3.

11. (a) $\log_{10} 4 = 2 \log_{10} 2 = 2r$

(b) $\log_{10} 6 = \log_{10} 2 + \log_{10} 3 = r + s$

(c) $\log_{10} \frac{1}{8} = -3 \log_{10} 2 = -3r$

(d) $\log_{10} 10 = r + t = 1$

(e) $\log_{10} 2.5 = \log_{10} 5 - \log_{10} 2 = t - r$

(f) $\log_{10} \frac{2}{9} = \log_{10} 2 - 2 \log_{10} 3 = r - 2s$

(g) $\log_{10} \frac{5}{9} \sqrt{3} = \log_{10} 5 - 2 \log_{10} 3 + \frac{1}{2} \log_{10} 3 = t - \frac{3}{2} s$

(h) $\log_{10} 8 \sqrt[3]{100} = 3 \log_{10} 2 + \frac{2}{3} (\log_{10} 2 + \log_{10} 5)$

$= 3r + \frac{2}{3} (r + t) = \frac{1}{3} (11r + 2t)$

12. (a) $\log_{10} 1000 = 3$

(b) $\log_{0.01} 0.001 = x \Leftrightarrow (0.01)^x = (10^{-2})^x = 10^{-3} \Leftrightarrow x = 1.5$

(c) $\log_3 (\frac{1}{81}) = \log_3 (3^{-4}) = -4$

(d) $\log_4 (32) = x \Leftrightarrow 4^x = 2^5 \Leftrightarrow x = 2.5$

(e) $\log_{10} (0.0001) = -4$ since $10^{-4} = 0.0001$

(f) $\log_{0.5} 16 = x \Leftrightarrow (\frac{1}{2})^x = 2^{-x} = 2^4 \Leftrightarrow x = -4$

(g) $\log_2 2^3 = 3$

(h) $\log_{10} \sqrt{10} = \frac{1}{2}$

(i) $\log_{81} 27 = x \Leftrightarrow (3^4)^x = 3^3 \Leftrightarrow x = \frac{3}{4}$

(j) $\log_2 \sqrt{32} = x \Leftrightarrow 2^x = (2^5)^{1/2} \Leftrightarrow x = \frac{5}{2}$

13. (a) $5 + 5 = x \Leftrightarrow x = 10$

(b) $\log_{10} \dfrac{x^2 - 1}{(x - 1)^2} = \log_{10} \dfrac{x + 1}{x - 1} = \log_{10} 3 \Leftrightarrow x + 1 = 3(x - 1)$, and $x = 2$

(c) $(\log_x 5)(\log_7 7) = \log_7 5 \Longleftrightarrow x = 7$

14. (a) 1

(b) $\frac{1}{10}$

(c) 10

(d) $10^3 = x - 2 \Longleftrightarrow x = 2 + 10^3$

(e) 10^{-3}

(f) $10^{-2} = 2x - 1 \Longleftrightarrow x = \dfrac{10^{-2} + 1}{2} = \dfrac{101}{200}$

15. (a) 1

(b) For all real $x > 0$.

(c) For all real $x > 0$ where $x \neq 1$.

(d) $\log_x 2^x = 2 \Leftrightarrow x^2 = 2^x \Leftrightarrow x = 2^{x/2}$

Roots are 2, 4, obtained by trial or by graphing $y = x$ and $y = 2^{x/2}$.

16. (a) Show that if $a > 1$, then $x \to \log_a x$ is an increasing function; i.e., show that for $a > 1$, if $x_1 < x_2$, then $\log_a x_1 < \log_a x_2$.

Given: $x_1 < x_2$. If $a > 1$, it follows that

$$a^{\log_a x_1} < a^{\log_a x_2}$$

and $$\log_a x_1 < \log_a x_2.$$

(b) Show that if $0 < a < 1$, then $x \to \log_a x$ is a decreasing function; i.e., show that for $0 < a < 1$, if $x_1 < x_2$, then $\log_a x_1 > \log_a x_2$.

Given: $x_1 < x_2$. If $0 < a < 1$, it follows that

$$a^{\log_a x_1} > a^{\log_a x_2}$$

and $$\log_a x_1 > \log_a x_2.$$

17. (a) In (4) $\log_a b = \dfrac{\log_c b}{\log_c a}$ substitute x for b, and b for c.

We then have $\log_a x = \dfrac{\log_b x}{\log_b a}$

$\therefore$ $\log_b x = (\log_b a)\log_a x$, or using (5):

$\log_a x = (\log_a b)\log_b x$.

i.e.: In the first relationship $\log_b a$ is the constant of proportionality; and in the second, $\log_a b$.

(b) $\log_a x = 0$, if $x = 1$. Therefore, the x-intercept is $(1,0)$.

18. $\log_a b = \dfrac{\log_c b}{\log_c a}$ (4). Let $c = b$, and we have $\log_a b = \dfrac{\log_b b}{\log_b a}$.

$\therefore$ $\log_a b = \dfrac{1}{\log_b a}$ (5).

Teacher's Commentary

Appendix 1

FUNCTIONS AND THEIR REPRESENTATIONS

TC A1-1. Functions

Functions whose domains and ranges are subsets of real numbers are usually called "real valued functions of a real variable."

Example A1-1b. The statement that the range of the functions $f : x \longrightarrow x^2$, is the set of nonnegative real numbers is equivalent to the statement that every positive number p has a square root. This can be proved in the following way. Let A be the set of nonnegative numbers whose squares are less than p. A is not empty since 0 is in A. A is bounded above, say by $p + 1$. Thus, A has a least upper bound, s (Appendix 7). We cannot have $s^2 < p$ for, if $h < 1$, then

$$\begin{aligned}(s + h)^2 &= s^2 + 2sh + h^2 \\ &= s^2 + h(2s + h) \\ &< s^2 + h(2s + 1) \\ &< s^2 + h(2p + 3).\end{aligned}$$

Thus, if

$$h < \min\{1, \frac{p - s^2}{2p + 3}\},$$

then $(s + h)^2 < p$ and s could not be an upper bound for A. Similarly, $s^2 > p$ is impossible, for if

$$0 < h < \frac{s^2 - p}{2s},$$

then $(s - h)^2 > p$, and thus s could not be the least upper bound for A. Hence $s^2 = p$.

Example A1-1e. We can define many other functions whose graphs are contained in the graph of $x^2 + y^2 = 25$; we need only assign to each x either $\sqrt{25 - x^2}$ or $-\sqrt{25 - x^2}$. For example, for any a, $-5 \le a < 5$, we have,

$$f_a : x \to \begin{cases} \sqrt{25 - x^2}, & -5 \le x \le a \\ -\sqrt{25 - x^2}, & a < x \le 5, \end{cases}$$

or

$$f_{unnny} : x \to \begin{cases} \sqrt{25 - x^2}, & x \text{ rational}, \quad |x| \le 5 \\ -\sqrt{25 - x^2}, & x \text{ irrational}, \quad |x| < 5. \end{cases}$$

However, the examples in the text are distinguished by the property that they are the only continuous functions defined on $[-5,5]$ whose graphs are contained in the graph of $x^2 + y^2 = 25$.

Solutions Exercises A1-1

1. Below are given examples of associations between elements of two sets. Decide whether each example may properly represent a function. This also requires you to specify the domain and range for each function. Note that no particular variable has to be the domain variable, and also some of the relations may give rise to several functions.

   Note that answers supplied are not necessarily the only correct ones. They are, like the examples, merely samples of the kind of ideas that are possible.

   (a) Assign to each nonnegative integer n the number $2n - 5$.

   This is a function with domain the set of nonnegative integers and range the set of odd integers not less than -5.

   (b) Assign to each real number x the number 7.

   A constant function. Domain the set of real numbers. Range the set consisting of the one element, 7.

   (c) Assign to the number 10 the real number y.

   Not a function.

(d) Assign to each pair of distinct points in the plane the distance between them.

A function whose domain is the set of all pairs of distinct points in the plane and whose range is the set of positive ~~real~~ numbers.

(e) $y = -3$ (for all x)

May represent a function (constant), whose domain is the set of real numbers. The range is $\{-3\}$.

(f) $x = 4$ (for all y and z).

Not a function if x is considered the domain variable. It is a function if the set of ordered pairs (y,z) is considered the domain with y and z real numbers. The range is $\{4\}$.

(g) $x + y = 2$

A function, domain = $\{x : x \text{ is a real number}\}$ and range = $\{y : y = 2 - x\}$, or vice versa.

(h) $y = 2x^2 + 3$

A function, with domain the set of all real numbers and range = $\{y : y \geq 3\}$.
If y is taken as the domain variable, the range set must be restricted to avoid ambiguity, and if the range is restricted to reals, the domain may have to be restricted to real numbers ≥ 3.

(i) $y^2 - 4 = x$

If x is an element of the domain, this equation does not define a function explicitly. But $f : x \to y = \sqrt{x + 4}$, $x \geq -4$, and $g : x \to y = -\sqrt{x + 4}$, $x \geq -4$, are functions whose ranges are the nonnegative and nonpositive real numbers, respectively. Also $h : y \to x = y^2 - 4$, $y \leq -2$ or $y \geq 2$, is a function whose range is $\{x : x \geq -4\}$.

(j) $y < 2x - 1$

Not a function.

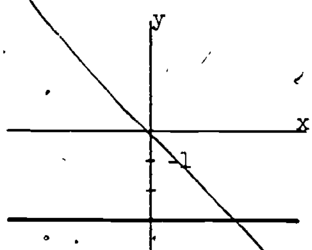
(k) $f(x) = -\sqrt{16 - x^2}$

A function. Usually, the domain is restricted to the set $\{x : -4 \leq x \leq 4\}$ so that the range will be real numbers, here the interval $[-4,0]$. Complex numbers are not used in this course in the calculus.

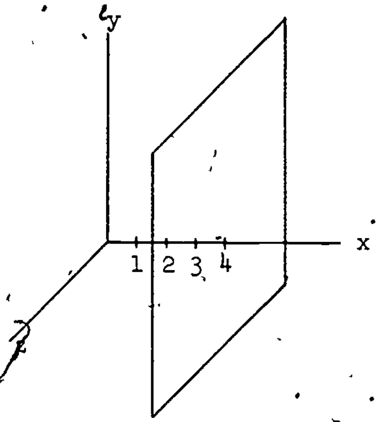
(ℓ) $x^2 + y^2 = 16$

This equation does not represent a function explicitly. See part (k) for one possible function obtained from this equation.

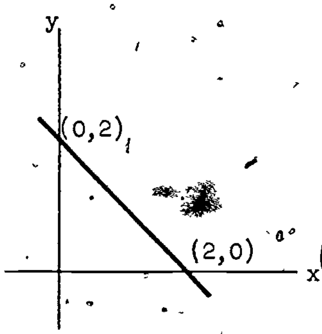
2. (e)



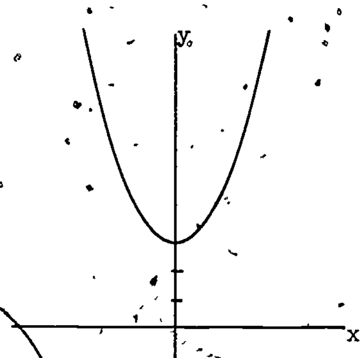
(f)



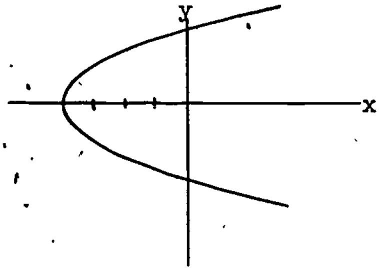
(g)



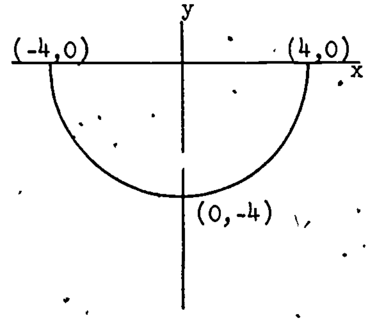
(h)



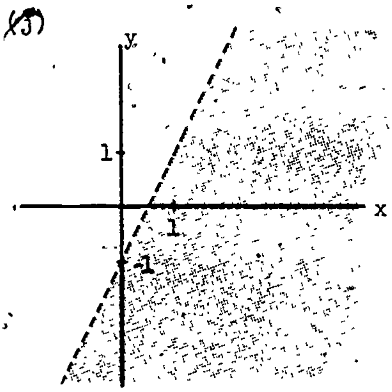
(i)



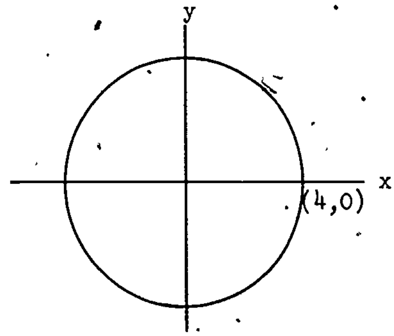
(k)



(j)



(ℓ)



3. A function f is completely defined by the table:

x	0	1	2	3	4
f(x)	-3	1	5	9	13

(a) Describe the domain and range of f.

Domain = {0, 1, 2, 3, 4}. Range = {-3, 1, 5, 9, 13}.

(b) Write an equation with suitably restricted domain that defines f.

$f(x) = 4x - 3$, x, an integer, and $0 \leq x \leq 4$.

4. If $f : x \rightarrow x^2 + 3x - 4$, find

(a) $f(0) = -4$.

(b) $f(2) = 6$.

(c) $f(-1) = -6$.

(d) $f(\sqrt{3}) = 3\sqrt{3} - 1$.

(e) $f(2 - \sqrt{2}) = 8 - 7\sqrt{2}$.

(f) $ff(1) = f(0) = -4$.

5. If g is a function defined by $g(x) = \dfrac{2x}{\sqrt{5 - x^2}}$, find if possible:

(a) $g(0) = 0$

(b) $g(1) = 1$

(c) $g(-1) = -1$

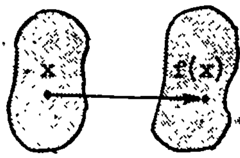
(d) $g(2) = 4$

(e) $g(-3)$ } undefined.

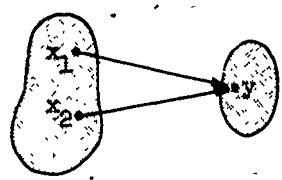
(f) $g(\sqrt{5})$ }

6. Which of the following mappings represent functions?

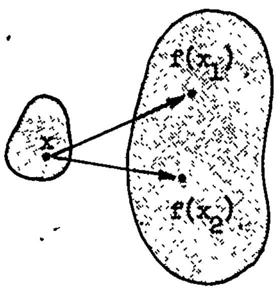
(a)



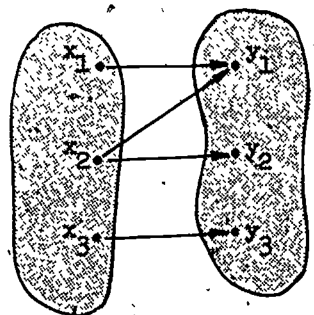
(d)



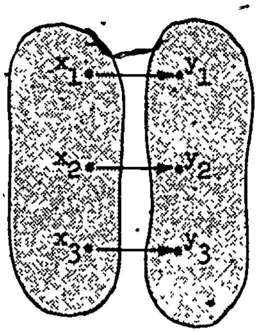
(b)



(e)



(c)



(a), (c) and (d) are functions.

7. Given the function $f : x \longrightarrow x$ and $g : x \longrightarrow \dfrac{x^2}{x}$. If x is a real number, are f and g the same function? Why or why not?

They are NOT the same function, since $x = 0$ is not in the domain of g but is in f.

8. Given the functions $f : x \rightarrow x + 2$ and $g : x \rightarrow \frac{x^2 - 4}{x - 2}$. If x is real, are f and g the same function? Why or why not?

They are not the same function, since $x = 2$ is not in the domain of g but is in f.

9. What number or numbers have the image 10 under the following mappings?

(a) $f : x \rightarrow 2x$	$x = 5$
(b) $g : x \rightarrow x^2$	$x = \pm\sqrt{10}$
(c) $h : x \rightarrow \sqrt{x^2 + 36}$	$x = \pm 8$
(d) $\alpha : x \rightarrow \lvert x - 4\rvert$	$x = 14, -6$
(e) $\phi : x \rightarrow [x]$	$10 \leq x < 11$

10. Which of the following statements are always true for any function f, assuming that x_1 and x_2 are in the domain of f?

(a) If $x_1 = x_2$, then $f(x_1) = f(x_2)$.

(b) If $x_1 \neq x_2$, then $f(x_1) \neq f(x_2)$.

(c) If $f(x_1) = f(x_2)$, then $x_1 = x_2$.

(d) If $f(x_1) \neq f(x_2)$, then $x_1 \neq x_2$.

(a) and (d) are always true.

11. If $f(x) = |x|$, which of the following statements are true for all real numbers x and t?

(a) f is an odd function.

(b) $f(x^2) = \left(f(x)\right)^2$

(c) $f(x - t) \leq f(x) - f(t)$

(d) $f(x + t) \leq f(x) + f(t)$

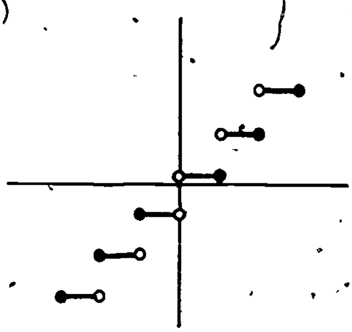
(b) and (d) are true statements.

12. Which of the following functions are even, which are odd, and which are neither even nor odd?

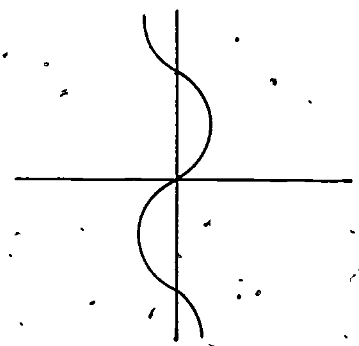
(a)	$f : x \to 3x$	odd
(b)	$f : x \to -2x^2 + 5$	even
(c)	$f : x \to x^2 - 4x + 4$	neither
(d)	$f : x \to -2x + 1$	neither
(e)	$f : x \to x^3 + 4$	neither
(f)	$f : x \to x^3 - 2x$	odd
(g)	$f : x \to 2^{1/x}$	neither
(h)	$f : x \to 2^{1/x^2}$	even

13. Which of the following graphs could represent functions?

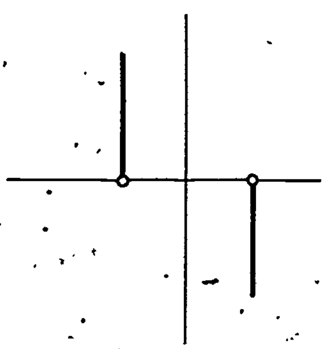
(a)



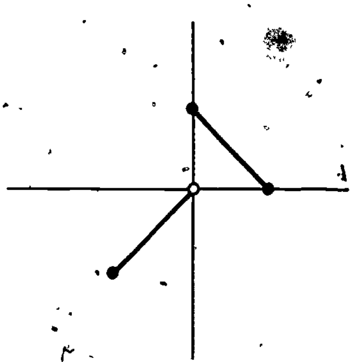
(c)



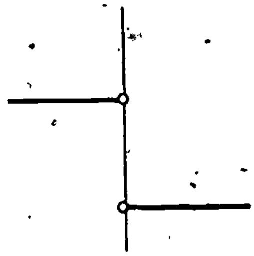
(b)



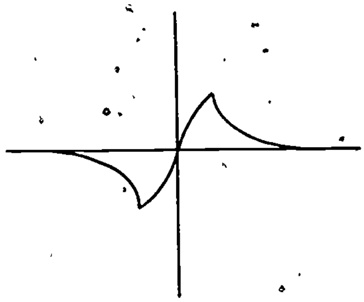
(d)



(e)

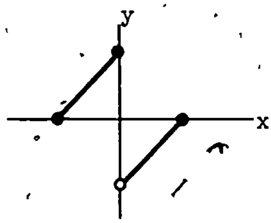


(f)



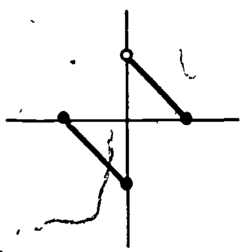
a, d, e and f could represent functions.

14. Suppose that $f : x \rightarrow f(x)$ is the function whose graph is shown.

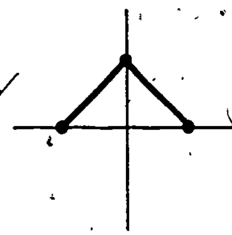


Sketch the graphs of

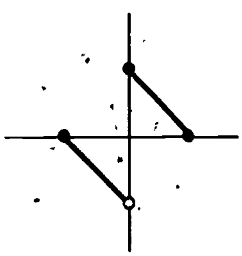
(a) $g : x \rightarrow -f(x)$



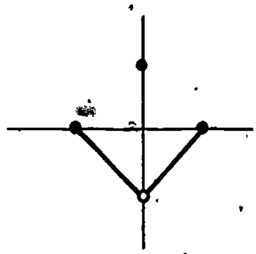
(c) $g : x \rightarrow |f(x)|$



(b) $g : x \rightarrow f(-x)$



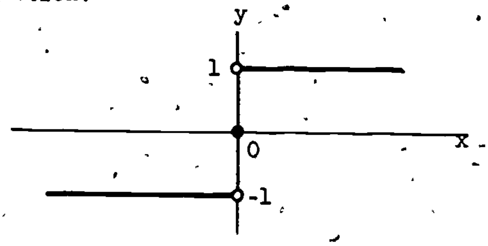
(d) $g : x \rightarrow f(|x|)$



15. A function f is defined by $f(x) = \begin{cases} \frac{x}{|x|} & \text{for } x \neq 0, \\ 0 & \text{for } x = 0. \end{cases}$

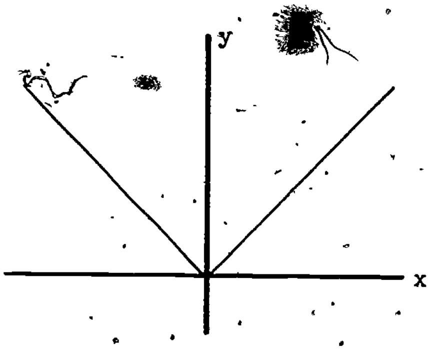
Identify this function and sketch its graph.

The signum function.



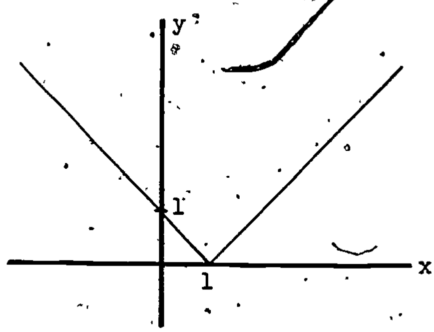
16. Sketch the graph of each function, specifying its domain and range. In each case, the domain is the set of all real numbers.

(a) $f : x \to \sqrt{x^2}$



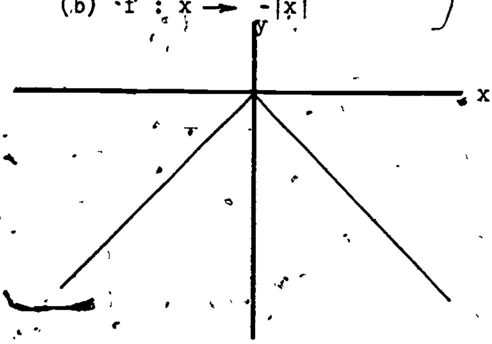
Range: all nonnegative reals.

(c) $f(x) = |1 - x|$



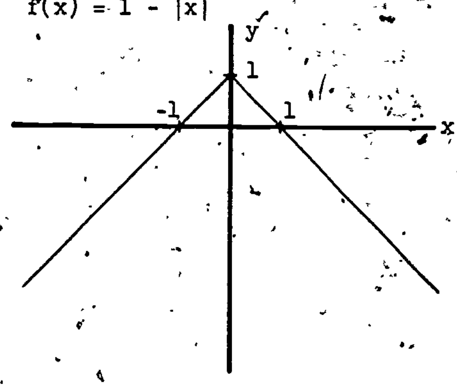
Range: all nonnegative reals.

(b) $f : x \to -|x|$



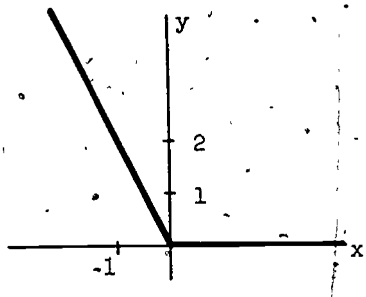
Range: all nonpositive reals.

(d) $f(x) = 1 - |x|$



Range = $\{y : y \leq 1\}$

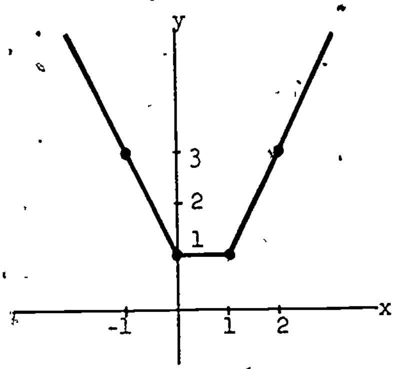
(e) $f : x \to |x| - x$



Range: all nonnegative reals.

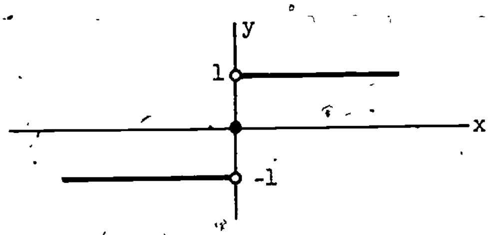
(f) $f : x \to |x| + |x - 1|$

(Hint: Consider separately the three possibilities: $x < 0$, $0 \le x \le 1$, and $x > 1$.)



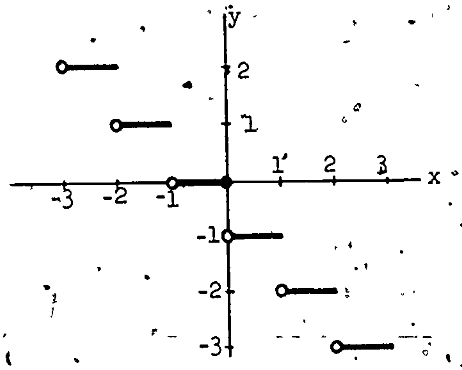
Range = $\{y : y \ge 1\}$

(g) $f : x \to \text{sgn } x$



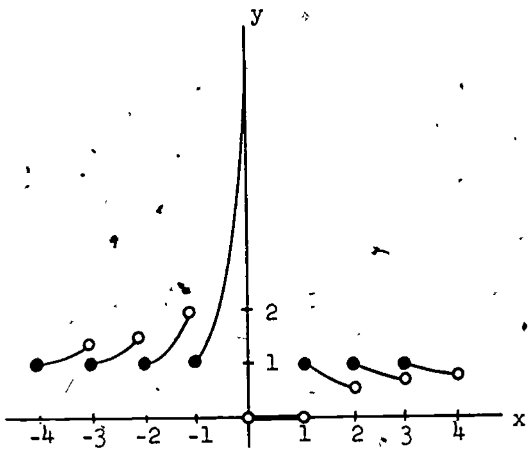
Range = $\{-1, 0, 1\}$

(h) $f : x \to [x]$



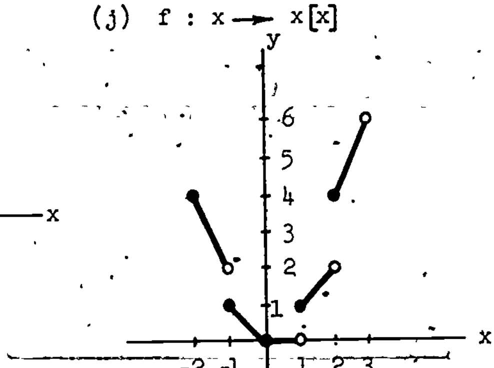
Range: all integers.

(i) $f : x \to \dfrac{[x]}{x}$



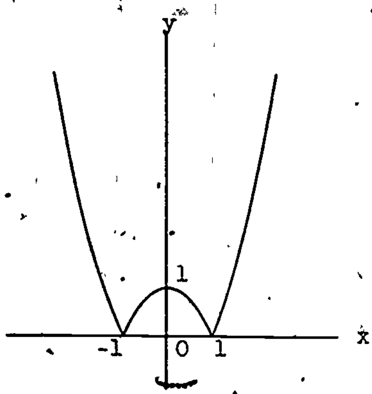
Range = $\{y : y = 0 \text{ or } y > \frac{1}{2}\}$

(j) $f : x \to x[x]$



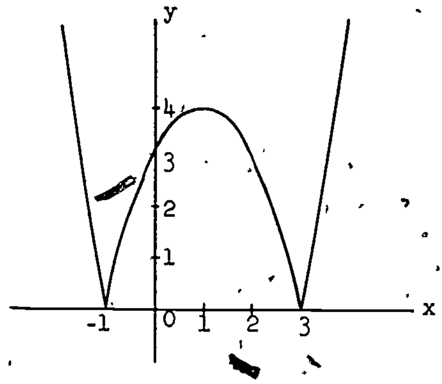
Range = $\{y : y \ge 0$ except $y = n^2 - n$, where n is an integer $\ge 2\}$.

(k) $f : x \to |1 - x^2|$



Range = $\{y : y \geq 0\}$

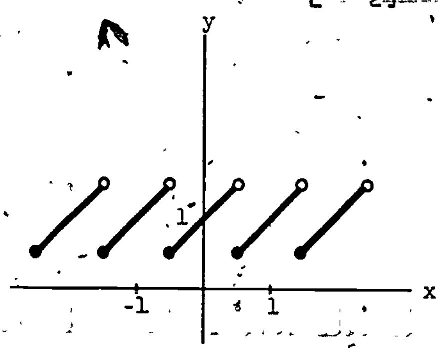
(ℓ) $f : x \to |x^2 - 2x - 3|$



Range = $\{y : y \geq 0\}$

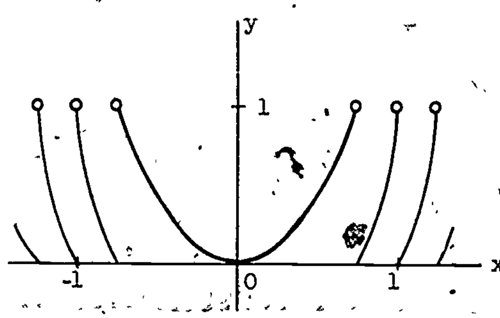
Sketch the graphs of the functions in Exercises 17 to 19. For those functions which are periodic, indicate their period. Indicate those functions which are even or odd.

17. (a) $f : x \to x - \left[x - \frac{1}{2}\right]$



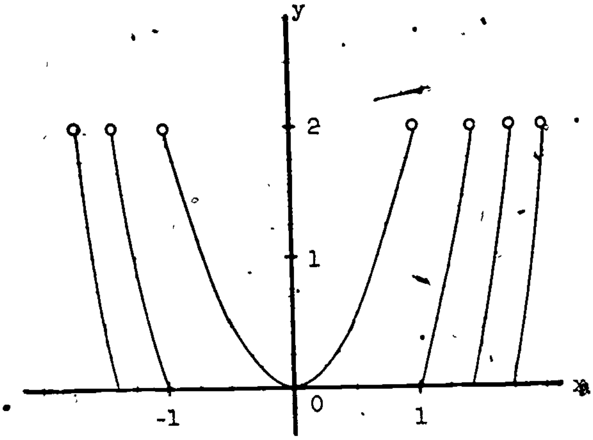
Periodic. $p = 1$

(b) $f : x \to 2x^2 - \left[2x^2\right]$

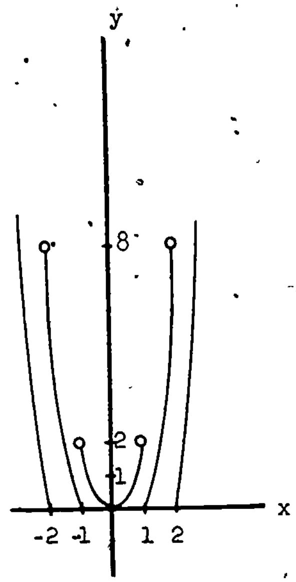


Function is EVEN.

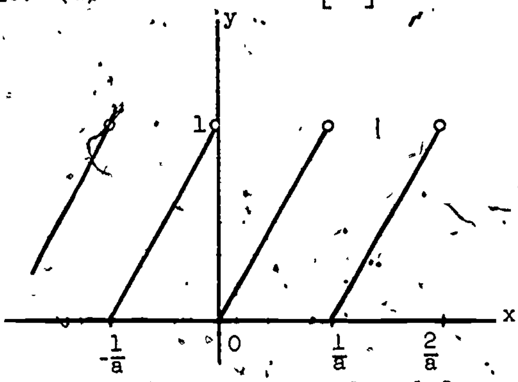
(c) $f : x \to 2x^2 - 2[x^2]$



(d) $f : x \to 2x^2 - 2[x]^2$



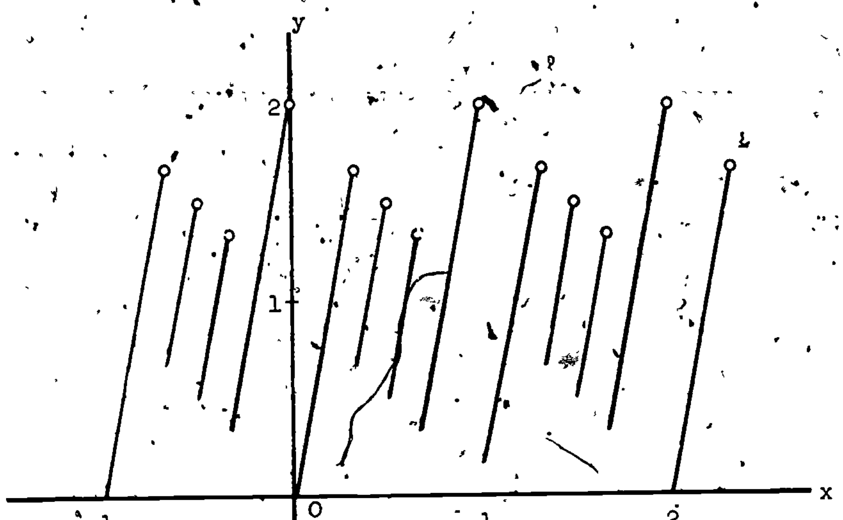
18. (a) $f : x \to a - [ax]$, $a > 0$



Periodic: $p = \frac{1}{a}$

Note that the slope of each piece is a.

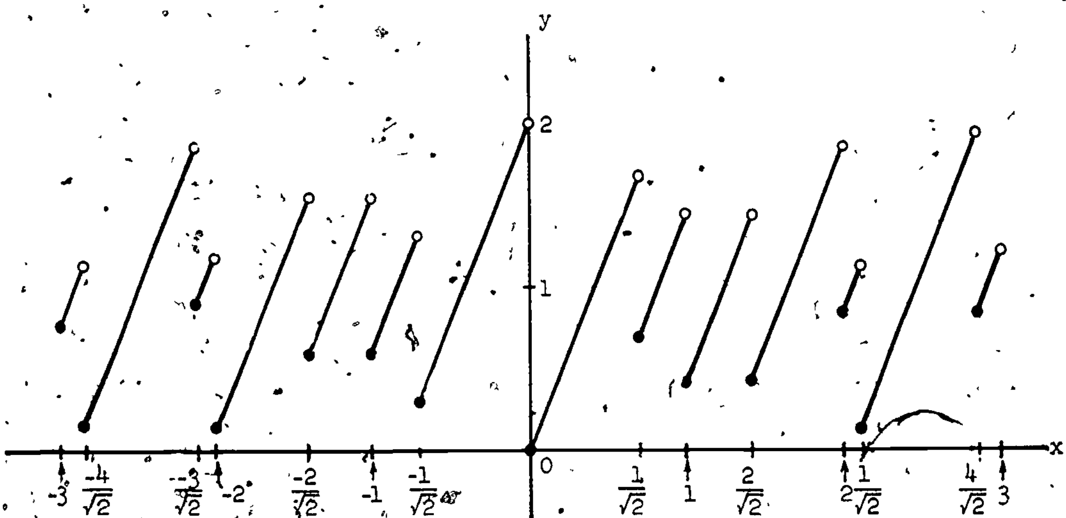
(b) $f : x \to 5x - [2x] - [3x]$



Note that $f(x) = g(x) + h(x)$ where $g(x) = 2x - [2x]$ and $h(x) = 3x - [3x]$ two periodic functions like that of part (a). Functions g and h have periods $\frac{1}{2}$ and $\frac{1}{3}$, respectively; hence $g + h$ is periodic and its period is the least common integer multiple of $\frac{1}{2}$ and $\frac{1}{3}$, which is 1; i.e., $2 \cdot \frac{1}{2} = 3 \cdot \frac{1}{3} = 1$. In general, if two periodic functions f_1 and f_2 have periods p_1 and p_2, respectively (p_1, p_2, positive real numbers), and if there exist integers n_1 and n_2 such that $n_1p_1 = n_2p_2$, then $f_1 + f_2$ is periodic with period n_1p_1.

Note also that the slope of each piece of the graph of f is 5.

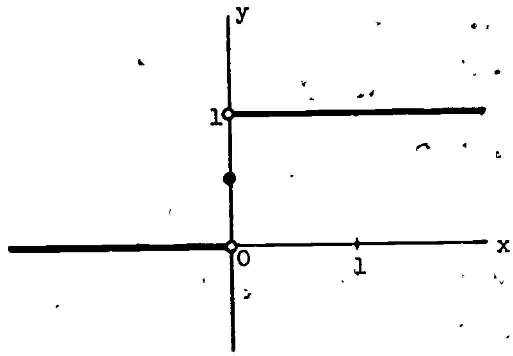
(c) $f : x \rightarrow x(\sqrt{2} + 1) - [x\sqrt{2}] - [x]$



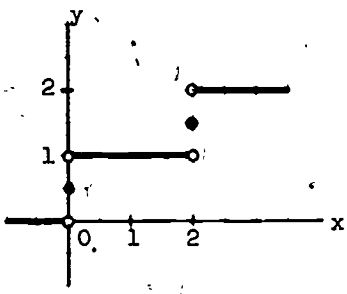
In this case, $f(x) = g(x) + h(x)$ where $g(x) = x\sqrt{2} - [x\sqrt{2}]$ and $h(x) = x - [x]$. The periods of g and h are $\frac{3}{\sqrt{2}}$ and 1 respectively, and since these numbers are incommensurable, $f = g + h$ is not a periodic function. Note that the points of discontinuity occur at the integers and at integer multiples of $\frac{1}{\sqrt{2}}$. The slope of each piece of the graph of f is $1 + \sqrt{2}$.

19. (a) $f : x \to \frac{1 + \text{sgn } x}{2}$

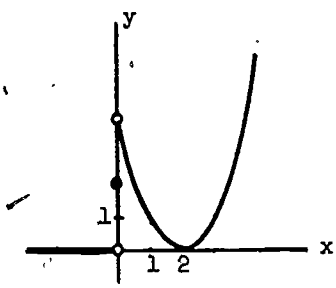
This function is also called the Heaviside Unit function and is designated by $f : x \to H(x)$.



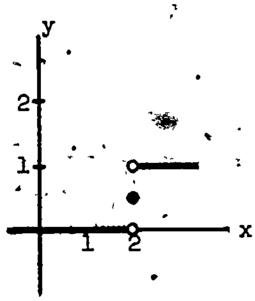
(b) $f : x \to H(x) + H(x - 2)$



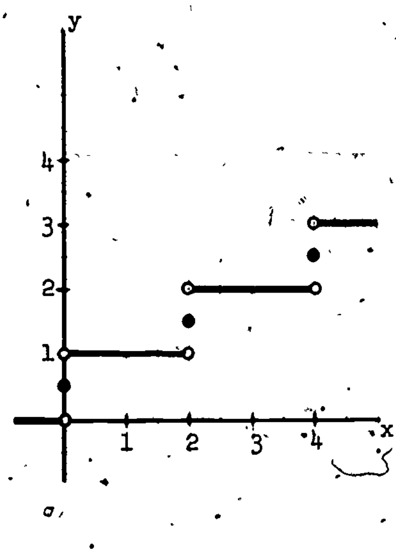
(d) $f : x \to (x - 2)^2 \cdot H(x)$



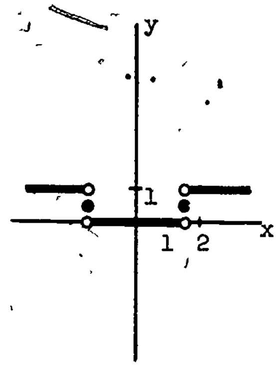
(c) $f : x \to H(x) \cdot H(x - 2)$



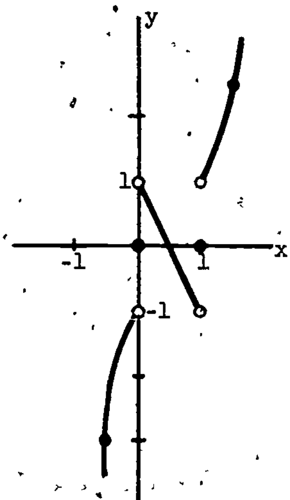
(e) $f : x \to H(x) + H(x - 2) + H(x - 4)$



(f) $f : x \longrightarrow H(x^2 - 2)$



(g) $f : x \longrightarrow (\text{sgn } x)(x - 1)^2 + [\text{sgn}(x - 1)]\, x^2$



20. If f and g are periodic functions of periods m and n, respectively, (m and n, integers), show that $f + g$, and $f \cdot g$ are also periodic. Give examples to show that the period of $f + g$ can either be greater or less than both of m and n. Repeat the same for the product $f \cdot g$. (Note: The problem in the text lacks the condition. "m and n, integers.")

Since any integer multiple of a period is also a period, $f + g$ and $f \cdot g$ are periodic with period mn. However, this may or may not be the fundamental period.

If we choose $f(x) = \sin \pi x$ (period 2) and $g(x) = \sin \frac{2\pi x}{3}$ (period 3), then $f + g$ and $f \cdot g$ both have period 6.

If, however, we choose $f(x) = \sin \frac{2\pi x}{3}$ (period 3) and $g(x) = \sin \pi x - \sin \frac{2\pi x}{3}$ (period 6), then $f + g$ has period 2.

If $f(x) = 2 \sin \pi x$ (period 2) and $g(x) = \cos \pi x$ (period 2), then $f(x) \cdot g(x) = \sin 2\pi x$ which has period 1.

21. (a) Can a function be both even and odd?

Yes; only $f : x \to 0$.

(b) What can you say about the evenness or oddness of the product of:

(1) an even function by an even function? EVEN
(2) an even function by an odd function? ODD
(3) an odd function by an odd function? EVEN

(c) Show that every function whose domain contains $-x$ whenever it contains x can be expressed as the sum of an even function plus an odd function.

$$f(x) = \underbrace{\frac{f(x) + f(-x)}{2}}_{\text{EVEN}} + \underbrace{\frac{f(x) - f(-x)}{2}}_{\text{ODD}}$$

22. Find functions $f(x)$ satisfying $f(x) \cdot f(-x) = 1$ (called a functional equation).
Suggestion: Use Number 21(c).

Let $f(x) = E(x) + \theta(x)$ where E is even and θ is odd. Then $f(-x) = E(x) - \theta(x)$. Using the given condition, we have $f(x) \cdot f(-x) = E^2(x) - \theta^2(x) = 1$. Hence, $E(x) = \sqrt{1 + \theta^2(x)}$ and $f(x) = \sqrt{1 + \theta^2(x)} + \theta(x)$, where θ is an arbitrary odd function. For example,

$$\theta(x) = x \quad , \quad f(x) = \sqrt{1 + x^2} + x;$$
$$\theta(x) = \tan x \; , \quad f(x) = \sec x + \tan x.$$

Alternatively. Let $f(0) = 1$ or -1 and let $f(x)$ be an arbitrary nonzero function for $x > 0$. Then define $f(-x) = \frac{1}{f(x)}$ for $x > 0$. This construction gives the entire class of functions satisfying the given functional equation.

23. Prove that no periodic function other than a constant can be a rational function. (Note: A rational function is the ratio of two polynomial functions.)

Our proof is indirect. Assume that there exists a rational function $\frac{P(x)}{Q(x)}$ (P, Q, polynomials) which has a period $m > 0$. Let k denote the value of the function at $x = a$ where $Q(a) \neq 0$. Now consider the polynomial equation $P(x) - kQ(x) = 0$. This polynomial vanishes for $x = a + nm$, $n = 0, 1, 2, \ldots$. Since a polynomial of degree d can have at most d roots, we get a contradiction, and thus our initial assumption is false.

TC A1-2. Composite Functions

In the introduction to this section it is suggested that the operation of composition for functions has no counterpart in the algebra of numbers. However, the set $P(A)$, consisting of all one-to-one mappings of the set A onto itself, with the operation of composition has the following properties (we denote the inverse of f by f^{-1} and the identity function by I):

(i) If $g, h, \in P(A)$, then $gh \in P(A)$.

(ii) $(fg)h = f(gh)$ for all $f, g, h, \in P(A)$.

(iii) $fI = If = f$ for all $f \in P(A)$.

(iv) $ff^{-1} = f^{-1}f = I$ for all $f \in P(A)$.

These are the postulates for a group.* Thus, $P(A)$ shares the same algebraic structure with the positive reals under multiplication, with the reals under addition, and many other familiar groups.

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*Group Postulates:
(i) A non-empty set having closure with respect to an operation.
(ii) Associativity.
(iii) The existence of an identity element.
(iv) The existence of an inverse element.

Solutions Exercises A1-2

1. Given that $f : x \to x - 2$ and $g : x \to x^2 + 1$ for all real x, find

   (a) $f(2) + g(2) = 5.$

   (b) $f(2) \cdot g(2) = 0.$

   (c) $fg(2) = 3.$

   (d) $gf(2) = 1.$

   (e) $f(x) + g(x) = x^2 + x - 1.$

   (f) $f(x) \cdot g(x) = x^3 - 2x^2 + x - 2.$

   (g) $fg(x) = x^2 - 1.$

   (h) $gf(x) = x^2 - 4x + 5.$

2. If $f(x) = 3x + 2$ and $g(x) = 5$, find

   (a) $fg(x) = f(5) = 17.$

   (b) $gf(x) = g(3x + 2) = 5.$

3. If $f(x) = 2x + 1$ and $g(x) = x^2$, find $fg(x)$ and $gf(x)$. For what values of x, if any, are $fg(x)$ and $gf(x)$ equal?

   $fg(x) = 2x^2 + 1;\ gf(x) = 4x^2 + 4x + 1.$

   $fg = gf$ when $2x^2 + 1 = 4x^2 + 4x + 1$, or $2x^2 + 4x = 0$, or

   $x = 0$, or -2.

4. For each pair of functions f and g, find the composite functions fg and gf, and specify the domain (and range, if possible) of each.

   (a) $f : x \to \frac{1}{x}$; $g : x \to 2x - 6$

   (b) $f : x \to \frac{1}{x}$; $g : x \to x^2 - 4$

   (c) $f : x \to \frac{1}{x}$; $g : x \to \sqrt{x}$

   (d) $f : x \to x^2$; $g : x \to \sqrt{x}$

   (e) $f : x \to x^2$; $g : x \to \sqrt{4 - x}$

   (f) $f : x \to -x^2$; $g : x \to \sqrt{x}$

	fg	Domain	Range	gf	Domain	Range
(a)	$\frac{1}{2x - 6}$	$x \neq 3$	$y \neq 0$	$\frac{2}{x} - 6$	$x \neq 0$	$y \neq -6$
(b)	$\frac{1}{x^2 - 4}$	$x \neq 2, -2$	$y \neq 0$	$\frac{1}{x^2} - 4$	$x \neq 0$	$y \neq -4$
(c)	$\frac{1}{\sqrt{x}}$	$x > 0$	$y > 0$	$\sqrt{\frac{1}{x}}$	$x > 0$	$y > 0$
(d)	x	$x \geq 0$	$y \geq 0$	$\|x\|$	all x	$y \geq 0$
(e)	$4 - x$	$x \leq 4$	$y \geq 0$	$\sqrt{4 - x^2}$	$[-2,2]$	$[0,2]$
(f)	$-x - 1$	$x \geq 0$	$y \leq -1$	Not a real function		

5. Given that $f(x) = x^2 + 3$ and $g(x) = \sqrt{x + 2}$, solve the equation

$$fg(x) = gf(x).$$

$fg(x) = |x + 2| + 3 = \sqrt{x^2 + 5} = gf(x)$ if and only if

$x^2 + 4x + 4 + 6|x + 2| + 9 = x^2 + 5$ or $4x + 8 + 6|x + 2| = 0$, whence $x = -2$.

<u>Note:</u> The use of "if and only if" requires absolute value.

<u>Alternatively</u>. Solve $x + 5 \in \sqrt{x^2 + 5}$ for the set of possible solutions and then check the results in the original equation.

6. Solve Problem 5 taking $g(x) = \sqrt{x - 2}$.

No solutions.

7. Describe functions f and g such that gf will equal:

(a) $3(x + 2) - 4$ $\qquad f(x) = x + 2 \quad ; \; g(x) = 3x - 4$

(b) $(2x - 5)^3$ $\qquad f(x) = 2x - 5 \quad ; \; g(x) = x^3$

(c) $\frac{3}{2x - 5}$ $\qquad f(x) = 2x - 5 \quad ; \; g(x) = \frac{3}{x}$

(d) $\sqrt{x^2 - 4}$ $\qquad f(x) = x^2 - 4 \quad ; \; g(x) = \sqrt{x}$

(e) $(x^4)^2$ $\qquad f(x) = x^4 \quad ; \; g(x) = x^2$

The answer is not unique. A simple answer may be $f(x) = x$ and $g(x)$ equal to function shown. The functions shown above are typical of those used later.

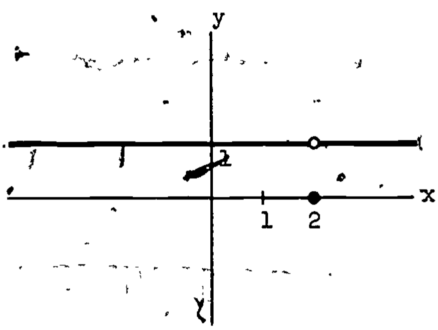
8. For each pair of functions f and g find the composite functions fg and gf and specify the domain (and range, if possible) of each. Also, sketch the graph of each, and give the period (fundamental) of those which are periodic.

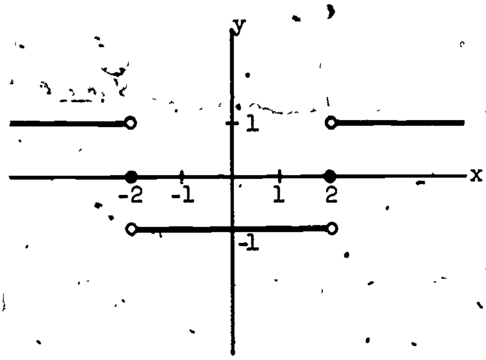
(a) $f : x \longrightarrow |x|$; $g : x \longrightarrow \operatorname{sgn}(x - 2)$

$fg(x) = |\operatorname{sgn}(x - 2)|$ $\qquad\qquad gf(x) = \operatorname{sgn}(|x| - 2)$

Domain: all reals; $\qquad$ Domain: all reals;

Range: $\{0,1\}$ $\qquad$ Range: $\{-1,0,1\}$



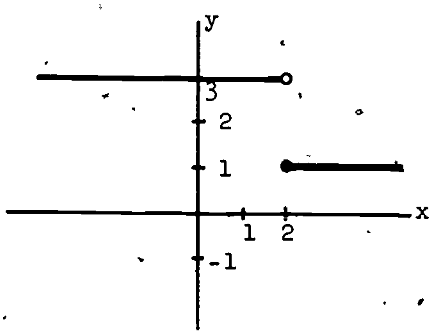


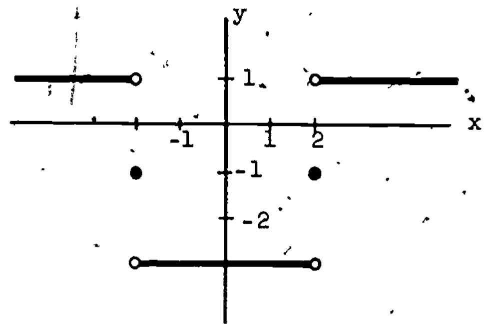
(b) $f : x \to |x|$; $g : x \to 2\text{sgn}(x - 2) - 1$

$fg(x) = |2\,\text{sgn}(x - 2) - 1|$ $\qquad$ $gf(x) = 2\text{sgn}(|x| - 2) - 1$

Domain: all reals; $\qquad$ Domain: all reals;

Range: $\{1,3\}$ $\qquad$ Range: $\{1,-1,-3\}$





9. What can you say about the evenness or oddness of the composite of an

(a)	even function of an even function?	EVEN
(b)	even function of an odd function?	EVEN
(c)	odd function of an odd function?	ODD
(d)	odd function of an even function?	EVEN

10. If the function f is periodic, what can you say about the periodic character of the composite functions fg and gf assuming these exist. g is an arbitrary function (not periodic). Illustrate by examples.

gf has the same periods as f and may have a smaller fundamental period.

fg need not be periodic.

Example: $\qquad f : x \to \sin x$, $\qquad$ period 2π;

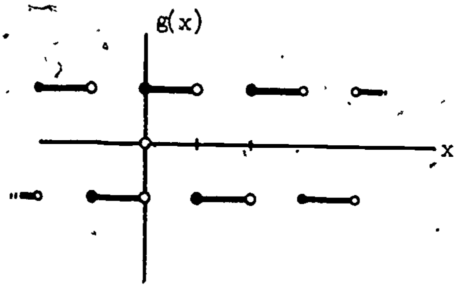
$g : x \to x^2$.

$gf : x \to (\sin x)^2$, $\qquad$ period π;

$fg : x \to \sin x^2$.

11. If the functions f and g are each periodic, then the composite functions fg and gf (assumed to exist) are also periodic. Can the period of either one be less than that of both f and g?

Yes. If $f(x) = \sin \pi x$, and if g is defined by



$$g(x) = \begin{cases} 1, & 0 \leq x < 1 \\ -1, & 1 \leq x < 2 \end{cases}$$

and $g(x + 2) = g(x)$, (g is called the square wave function). Then $fg(x) = 0$ for all x, and fg has any period $p > 0$. The period of gf is 2.

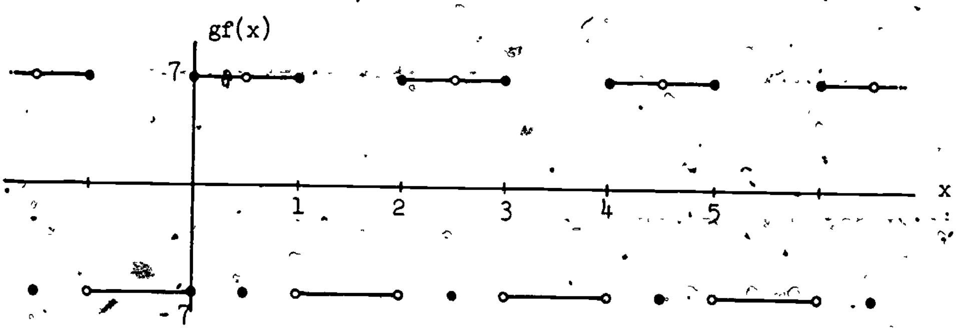
Since $g(x + 2) = g(x) = g(x - 2)$

$$g(x) = \begin{cases} -1, & -1 \leq x < 0 \\ 1, & 0 \leq x < 1 \\ -1, & 1 \leq x < 2 \end{cases}$$

and

$$g(\sin \pi x) = \begin{cases} -1, & -1 \leq \sin \pi x < 0 \\ 1, & 0 \leq \sin \pi x < 1 \\ -1, & 1 \leq \sin \pi x < 2 \end{cases}$$

and gf has period 2.



12. A sequence $a_0, a_1, a_2, \ldots, a_n, \ldots$ is defined by the equation

$$a_{n+1} = f(a_n), \quad n = 0, 1, 2, 3, \ldots$$

where f is a given function and a_0 is a given number. If $a_0 = 0$ and $f : x \rightarrow \sqrt{2 + x}$, then

$$a_1 = f(a_0) = \sqrt{2},$$

$$a_2 = f(a_1) = ff(a_0) = \sqrt{2 + \sqrt{2}},$$

$$a_3 = f(a_2) = ff(a_1) = fff(a_0) = \sqrt{2 + \sqrt{2 + \sqrt{2}}}.$$

Show that for any n,

(a) $a_n < 2$.

By Induction:

For $n = 1$, $\quad a_1 = \sqrt{2 + a_0} = \sqrt{2} < 2$.

Assume $\quad a_n < 2$ for $n = k$.

Then $\quad a_{k+1} = \sqrt{2 + a_k} < \sqrt{2 + 2}$

$$< \sqrt{4}$$

$$< 2 \qquad \text{Q.E.D.}$$

(b) $a_n > 2 - \dfrac{1}{2^{n-1}}$, $n > 0$.

The statement is true for $n = 1$. Suppose it is true for $n = k$.

Then $\quad a_{n+k} = \sqrt{2 + a_k}$

$$(a_{k+1})^2 - 4 = a_k - 2$$

$$2 - a_{k+1} = \frac{2 - a_k}{2 + a_{k+1}} < \frac{2 - a_k}{2}$$

$$< \frac{2 - (2 - \frac{1}{2^{k-1}})}{2}$$

$$\leq \frac{1}{2^k},$$

from which the conclusion follows.

13. If $a_{n+1} = f(a_n)$, $n = 0, 1, 2, \ldots$, $a_0 = \mu$, find a_n as a function of μ and n, for the following functions f:

(a) $f : x \rightarrow a + bx$

$$a_0 = \mu$$
$$a_1 = f(a_0) = a + b\mu$$
$$a_2 = f(a_1) = a + b(a + b\mu)$$
$$\cdots$$
$$a_n = f(a_{n-1}) = a + ab + ab^2 + \ldots + ab^{n-1} + b^n\mu$$

(b) $f : x \rightarrow x^m$

$$a_0 = \mu$$
$$a_1 = \mu^m$$
$$a_2 = (\mu^m)^m = \mu^{(m^2)}$$
$$\cdots$$
$$a_n = (\mu^{m^{n-1}})^m = \mu^{(m^n)}$$

(c) $f : x \rightarrow \sqrt{|x|}$

$$a_0 = \mu$$
$$a_1 = \sqrt{|\mu|} = |\mu|^{1/2} > 0$$
$$a_2 = (|\mu|^{1/2})^{1/2} = |\mu|^{1/4}$$
$$\cdots$$
$$a_n = |\mu|^{1/2^n}$$

(d) $f : x \rightarrow \sqrt{1 - x^2}$

$$a_0 = \mu$$
$$a_1 = \sqrt{1 - \mu^2}$$
$$a_2 = \sqrt{1 - (1 - \mu^2)} = |\mu|$$
$$\cdots$$
$$a_{2n} = |\mu| \text{ and } a_{2n+1} = \sqrt{1 - \mu^2} \quad \text{for } n > 0.$$

(e) $f : x \longmapsto (1 - x)^{-1}$

$$a_0 = \mu$$

$$a_1 = \frac{1}{1 - \mu}$$

$$a_2 = \frac{1}{1 - \frac{1}{1 - \mu}} = \frac{1 - \mu}{-\mu} = 1 - \frac{1}{\mu}$$

$$a_3 = \frac{1}{1 - (1 - \frac{1}{\mu})} = \frac{1}{\frac{1}{\mu}} = \mu$$

$$\cdots$$

$$a_{3n} = \mu, \ a_{3n+1} = \frac{1}{1 - \mu}, \ a_{3n+2} = 1 - \frac{1}{\mu}.$$

TC A1-3. Inverse Functions

Given any set A, a relation R on A is a rule which permits us to say for any pair (x,y) in A whether or not x has or has not the relation R to y. If x has the relation R to y then we write $x R y$; if not, we write $x \not R y$. Thus for the relation $<$ we have $1 < 2$ while $2 \not< 1$. The graph of a relation is the set of ordered pairs (x,y) for which $x R y$. Thus, the graph of the relation $<$ is the set of points above the line $y = x$. Consider the relation $x R y$ if and only if $x^2 + y^2 = 25$. The graph of this relation is the circle of radius 5 with center at the origin.

We have defined the inverse of a function only when the function is one-to-one. Moreover, we have required that the domain of the inverse of f be the range of f. Thus, although $\sin(\arcsin x) = x$ for all x in the domain of arcsine, sine is not the inverse of arcsine (i.e., $\arcsin(\sin x) = x$ is not true for all x). For these reasons it is essential that the definition of a function carry with it the domain of the function. Thus the functions

$$f : x \to \sin x \qquad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

$$g : x \to \sin x \qquad x \in R$$

are distinct, even though we may (by a convenient abuse of language) call them both by the same name, sine.

The situation that we have just described applies generally. Given any function f there is a function g, defined on the range of f, satisfying $fg(x) = x$ for each x in the range of f. We simply require $g : x \to$ any preimage of x (one must exist since x is in the range of f). Other authors sometimes designate g as a right inverse of f and f as a left inverse of g. The one-to-oneness of f is equivalent to the requirement that f has both a right and left inverse or the requirement that f has a unique right inverse.

Solutions Exercises A1-3

1. What is the reflection of the line $y = f(x) = 3x$ in the line $y = x$? Write an equation defining the inverse of f.

$$x = 3y \qquad \text{or} \qquad y = \frac{x}{3}.$$

2. Which points are their own reflections in the line $y = x$? What is the graph of all such points?

Those points with ordinate equal to abscissa. The graph is the line $y = x$.

3. (a) Find the slope of the segment from (a,b) to (b,a), and prove that the segment is perpendicular to the line $y = x$.

Slope of line from (a,b) to (b,a) is

$$\frac{a - b}{b - a} = -1.$$

Slope of $y = x$ is 1.

Since product of slopes $= -1$, the lines are perpendicular.

(b) Prove that the segment from (a,b) to (b,a) is bisected by the line $y = x$.

Let the point of intersection be (x,x); then

$$\sqrt{(a - x)^2 + (b - x)^2} = \sqrt{(x - b)^2 + (x - a)^2}.$$

4. What is the reflection of $(1,1)$ in the line

(a) $x = 0$? $(-1,1)$

(b) $y = 0$? $(1,-1)$.

(c) $y = -x$? $(-1,-1)$

(d) $y = 2$? $(1,3)$.

(e) $x = -3$? $(-7,1)$

5. Describe any function or functions you can think of which are their own inverses.

Any function whose graph has $y = x$ as an axis of symmetry; e.g.,

$$f(x) = x, \quad c - x, \quad \frac{1}{x}, \quad -\frac{1}{x},$$

$$\sqrt{1 - x^2} \quad \text{for} \quad 0 \le x \le 1.$$

6. An equation or an expression (phrase) is said to be symmetric in x and y if the equations or the expressions remain unaltered by interchanging x and y; e.g., $x^2 + y^2 = 0$, $x^3 + y^3 - 3xy$, $|x - y| = |x + y|$, $x - xy + y$. It follows that graphs of symmetric equations are symmetric about the $y = x$ line.
Geometrically, we can consider the line $y = x$ behaving as a mirror, i.e., for any portion of the graph there must also be a portion which is the mirror image. The equation

$$x^4 + y^4 = a^4$$

is obviously symmetric with respect to the line $y = x$. What other axes of symmetry (mirror type) does it have?

The x and y axes and the line $y = -x$. (Note: The graph is not a circle.)

7. The expression

$$a + b + |a - b| + 2c + \left|a + b + |a - b| - 2c\right|$$

is obviously symmetric in a and b. Show that it is also symmetric in a and c.

We have to show that

$$a + b + |a - b| + 2c + \left|a + b + |a - b| - 2c\right| = c + b + |c - b| + 2a + \left|c + b + |c - b| - 2a\right|.$$

Consider the six possible order relations

(1) $a \ge b \ge c$,

(2) $a \ge c \ge b$,

(3) $b \ge c \ge a$, etc.

For (1) and (2) we have $4a = 4a$; for (3), $4b = 4b$; and similarly for the other cases.

Another solution follows by noting that the given expression is also $4 \max\{a,b,c\}$, which is obviously symmetric in all three letters a, b, c, as well as any pair.

8. Find the inverse of each function.

(a) $f : x \rightarrow 3x + 6$ $\qquad g : x \rightarrow \frac{x - 6}{3}$

(b) $f : x \rightarrow x^3 - 5$ $\qquad g : x \rightarrow (x + 5)^{1/3}$

(c) $f : x \rightarrow \frac{2}{x} - 3$ $\qquad g : x \rightarrow \frac{2}{x + 3}$

9. Which of the following functions have inverses? Describe each inverse by means of a graph or equation and give its domain and range.

(a) $f : x \rightarrow x^2$ $\qquad$ Has no inverse.

(b) $f : x \rightarrow \sqrt{x}$ $\qquad g : x \rightarrow x^2$ Domain and range: all nonnegative real numbers.

(c) $f : x \rightarrow |x|$ $\qquad$ Has no inverse.

(d) $f : x \rightarrow [x]$ $\qquad$ Has no inverse.

(e) $f : x \rightarrow x|x|$ $\qquad g : x \rightarrow \sqrt{|x|}\ \text{sgn}\ x$ Domain and range: all real numbers.

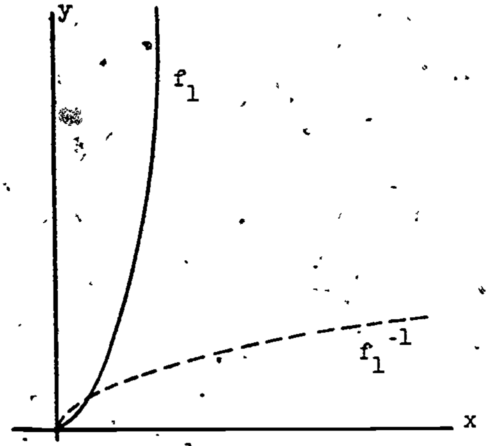
(f) $f : x \rightarrow \text{sgn}\ x$ $\qquad$ Has no inverse.

10. As we have seen, $f : x \rightarrow x^2$ for all real x does not have an inverse. Do the following:

(a) Sketch graphs of $f_1 : x \rightarrow x^2$ for $x \geq 0$ and $f_2 : x \rightarrow x^2$ for $x \leq 0$, and determine the inverses of f_1 and f_2.

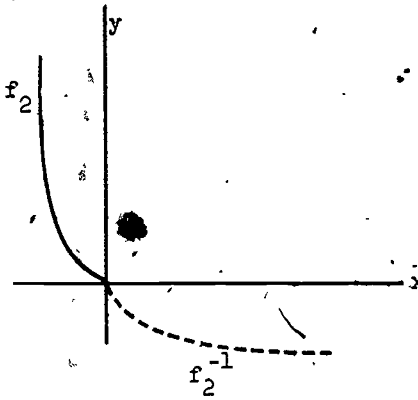
(b) What relationship exists among the domains of f, f_1, and f_2? (f_1 is called the <u>restriction</u> of f to the domain $\{x : x \geq 0\}$, and f_2 is similarly the restriction of f to the domain $\{x : x \leq 0\}$.)

(a)



$f_1^{-1}(x) = \sqrt{x}$

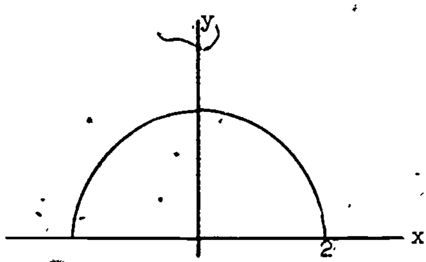
(b)



$f_2^{-1}(x) = -\sqrt{x}$

Domain of f = Domain of $f_1 \cup$ Domain of f_2

11. (a) Sketch a graph of $f : x \longrightarrow \sqrt{4 - x^2}$ and show that f does not have an inverse.



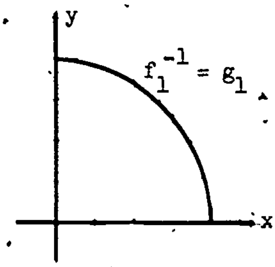
Each image (except 2) has two preimages; hence, f does not have an inverse.

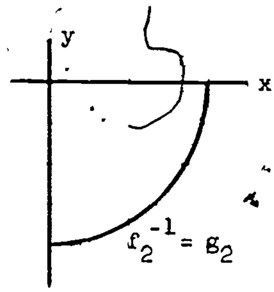
(b) Divide the domain of f into two parts such that the restriction of f to either part has an inverse.

$$\begin{cases} f_1 : x \longrightarrow \sqrt{4 - x^2} & \text{Domain} = \{x : 0 \le x \le 2\} \\ f_2 : x \longrightarrow \sqrt{4 - x^2} & \text{Domain} = \{x : -2 \le x \le 0\} \end{cases}$$

(c) Write an equation defining each inverse of part (b) and sketch the graphs.

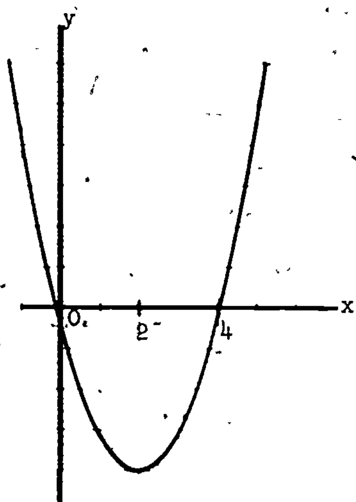
$$g_1 : x \longrightarrow \sqrt{4 - x^2} \qquad g_2 : x \longrightarrow -\sqrt{4 - x^2}$$





12. Do Problem 11 for $f : x \longrightarrow x^2 - 4x$.

(a)



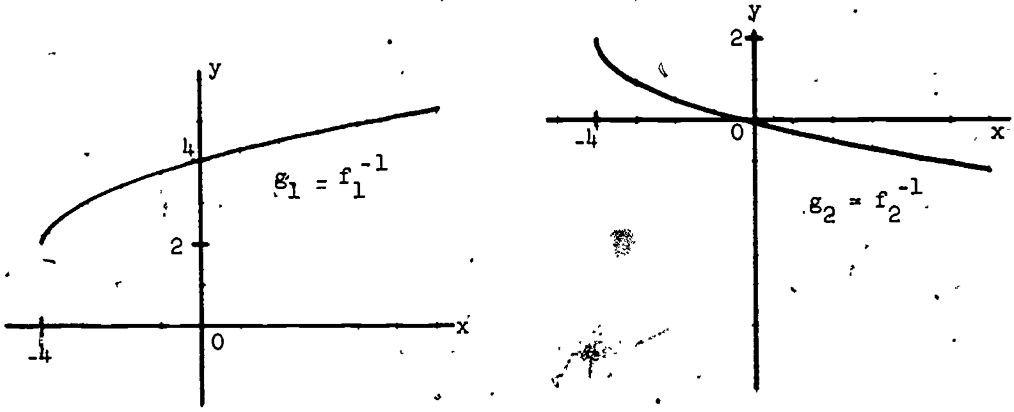
(b) $f_1 : x \longrightarrow x^2 - 4x$

Domain $\{x : x \geq 2\}$

$f_2 : x \longrightarrow x^2 - 4x$

Domain $\{x : x \leq 2\}$

(c) $g_1 : x \to 2 + \sqrt{4 + x}$ $\qquad$ $g_2 : x \to 2 - \sqrt{4 + x}$



13. Given that $f(x) = 3x - 2$ and $g(x) = -2x + k$, find k such that $fg(x) = gf(x)$. For this value of k are f and g inverse to one another? Give reasons for your answers.

No. $3(-2x + k) - 2 = -2(3x - 2) + k$ only for $k = 3$;

$$x \neq fg(x) = -6x + 7 = gf(x) = -6x + 7 \neq x.$$

The composition of two functions which are inverses is always the identity function, which is not the case here.

14. Show that $f : x \to x^2 - 4x + 5$ for $x \geq 2$ and $g : x \to 2 + \sqrt{x - 1}$ for $x \geq 1$ are inverse to one another by showing that $fg(y) = y$ for all y in the domain of g, and that $gf(x) = x$ for all x in the domain of f.

$$fg(y) = (2 + \sqrt{y - 1})^2 - 4(2 + \sqrt{y - 1}) + 5 = y, \quad y \geq 1;$$

$$gf(x) = 2 + \sqrt{x^2 - 4x + 5 - 1} = 2 + \sqrt{(x - 2)^2} = 2 + |x - 2| = x, \quad x \geq 2.$$

15. If $f(x) = (2x^3 + 1)^7$, find at least two functions g such that $fg(x) = gf(x)$.

One solution for g will be the inverse function of f or

$g(x) = \sqrt[3]{\left(\frac{\sqrt[7]{x} - 1}{2}\right)}$. Other simple solutions are $g(x) = x$ and $g(x) = k$(constant) where $k = f(k) = (2k^3 + 1)^7$. This equation has at least one negative real root.

TC A1-4. Monotone Functions

Example A1-4b. The comments (TC Example A2-1b) about the existence of a square root apply equally well to the existence of n-th roots. The same idea proves the existence of n-th roots. Thus, let A be the set of nonnegative numbers whose n-th power is less than p. Since $0 \in A$ and $p + 1$ is an upper bound for A, A has a least upper bound s. For $h < 1$

$$(s + h)^n = s^n + h \sum_{k=1}^{n} \binom{n}{k} s^{n-k} h^{k-1}$$

$$< s^n + h(n + 1)!s^n$$

$$< s^n + h(n + 1)!(p + 1)^n,$$

and

$$(s - h)^n = s^h + \sum_{k=1}^{n} \binom{n}{k} s^{n-k} h^k(-1)^k$$

$$> s^h - h \sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \binom{n}{k} s^{n-k} h^{k-1} (-1)^{k-1}$$

$$> s^n - h(n + 1)!(p + 1)^n.$$

Thus, as in TC Example A1-1b we can have neither $s^n < p$ nor $s^n > p$. Therefore $s^n = p$.

As stated in the text,

$$f : x \longrightarrow x^n, \quad n \text{ an odd natural number},$$

is a one-to-one function for all real x, and hence its inverse

$$g : y \longrightarrow \sqrt[n]{y}, \quad n \text{ an odd natural number},$$

is one-to-one for all real y. This raises notational problems. Since we write, for example, $\sqrt[3]{-27} = -3$, it is customary to write $\sqrt[3]{-y^3} = -y$ for all real y. In Chapter 5, however, fractional exponents are introduced, and we define $a^{1/n}$ only for $a > 0$. Thus we write $\sqrt[3]{y} = y^{1/3}$ only for $y > 0$.

One reason for this restriction is that it preserves the laws of exponents, in particular

$$(x^m)^n = x^{mn}.$$

Otherwise, the following ambiguity would arise:

$$[(-8)^{2/3}]^{1/2} = [-8]^{1/3} = -2,$$

and

$$[(-8)^{2/3}]^{1/2} = [4]^{1/2} = 2.$$

There is an alternative way of defining the inverse of

$$f : x \to x^3 = y, \quad \text{for all real } x \text{ and } y,$$

which you may wish to consider. If

$$x^3 = y,$$

then

$$x = (\operatorname{sgn} y)\sqrt[3]{|y|} = \begin{cases} y^{1/3} & \text{if } y \geq 0, \\ -|y|^{1/3} & \text{if } y < 0. \end{cases}$$

For example, if $x^3 = -8$, then

$$x = (-1)\sqrt[3]{|-8|} = -|-8|^{1/3} = -2.$$

We do not write $(-8)^{1/3}$.

Solutions Exercises A1-4

1. Prove that $f : x \to x^2$ for $x \geq 0$ is a strictly increasing function. (Hint: Let $x_1 > x_2 \geq 0$; then $x_1 - x_2 > 0$. From this show that $x_1^2 > x_2^2$.)

$$x_1 > x_2 \geq 0 \Rightarrow x_1 + x_2 > 0 \text{ and } x_1 - x_2 > 0 \Rightarrow x_1^2 - x_2^2 > 0$$
$$\Rightarrow x_1^2 > x_2^2.$$

2. Which of the following functions are decreasing? increasing?* decreasing? strictly increasing? In each case the domain is the set of real numbers unless otherwise restricted.

(a) $f_1 : x \to c$, c a constant — Increasing and decreasing.

(b) $f_2 : x \to x$ — Strictly increasing

(c) $f_3 : x \to |x|$ — Not monotone.

(d) $f_4 : x \to [x]$

(e) $f_5 : x \to \operatorname{sgn} x$

(d), (e): Increasing.

(f) $f_6 : x \to -x^2, \quad x \leq 0$ — Strictly increasing.

(g) $f_7 : x \to -\sqrt{x}, \quad x \geq 0$ — Strictly decreasing.

(h) $f_8 : x \to x|x|$ — Strictly increasing.

(i) $f_9 : x \to x + |x|$ — Increasing.

(j) $f_{10} : x \to |x| + |x - 1|$

(k) $f_{11} : x \to |x - 1| + |x - 3|$

(ℓ) $g_1 : x \to f_3 f_4(x)$

(m) $g_2 : x \to f_4 f_3(x)$

(j)–(m): Not monotone.

3. For each function in Problem 2 which is not monotone, divide its domain into parts such that the restriction of f to any of these parts gives a monotone or strictly monotone function.

(c) $x \leq 0$ and $x \geq 0$.

(j) $x \leq 0$ and $x \geq 0$ or $x \leq 1$ and $x \geq 1$.

(k) $x \leq 1$ and $x \geq 1$ or $x \leq 3$ and $x \geq 3$.

(ℓ) $x \leq 0$ and $x \geq 0$.

(m) $x \leq 0$ and $x \geq 0$.

---

* See footnote in A1-4.

4. We are given that the functions

f_1 is increasing,
f_2 is strictly increasing,
g_1 is decreasing,
g_2 is strictly decreasing,

in a common domain. What is the monotone character, if any, of the following functions;

(a) $f_1 + f_2$, Strictly increasing.

(b) $f_2 + g_1$, Not necessarily monotone.

(c) $g_1 + g_2$, Strictly decreasing.

(d) $g_2 + f_1$,
(e) $f_1 \cdot f_2$,
(f) $f_2 \cdot g_1$,
(g) $g_1 \cdot g_2$,
(h) $g_2 \cdot f_1$,

Not necessarily monotone.

(i) $f_1 f_2$,
(j) $f_2 f_1$,

Increasing.

(k) $f_2 g_1$,
(ℓ) $g_1 f_2$,

Decreasing.

(m) $g_1 g_2$,
(n) $g_2 g_1$,

Increasing.

(o) $g_2 f_1$,
(p) $f_1 g_2$,

Decreasing.

TC A1-5. Polar Coordinates

Descartes introduced oblique axes as well as perpendicular axes. The only reason that rectangular coordinate axes are preferred to oblique axes (at an angle θ, $0 < \theta < \frac{\pi}{2}$) is that the formula expressing the distance between points would become more complicated (by involving the familiar Law of Cosines).

Solutions Exercises A1-5

1. Find all polar coordinates of each of the following points:

   (a) $(6,\frac{\pi}{4}) = (6,\frac{\pi}{4} + 2n\pi),\ \left(-6,\frac{\pi}{4} + (2n + 1)\pi\right).$

   (b) $(-6,\frac{\pi}{4}) = (-6,\frac{\pi}{4} + 2n\pi),\ \left(6,\frac{\pi}{4} + (2n + 1)\pi\right).$

   (c) $(6,-\frac{\pi}{4}) = (6,-\frac{\pi}{4} + 2n\pi),\ \left(-6,-\frac{\pi}{4} + (2n + 1)\pi\right).$

   (d) $(-6,-\frac{\pi}{4}) = (-6,-\frac{\pi}{4} + 2n\pi),\ \left(6,-\frac{\pi}{4} + (2n + 1)\pi\right).$

2. Find rectangular coordinates of the points in Number 1.

   (a) $(6,\frac{\pi}{4}) = (3\sqrt{2}, 3\sqrt{2})$

   (b) $(-6,\frac{\pi}{4}) = (-3\sqrt{2}, -3\sqrt{2})$

   (c) $(6,-\frac{\pi}{4}) = (3\sqrt{2}, -3\sqrt{2})$

   (d) $(-6,-\frac{\pi}{4}) = (-3\sqrt{2}, 3\sqrt{2})$

3. Find polar coordinates of each of the following points given in rectangular coordinates:

   (a) $(4,-4) = (4\sqrt{2}, -\frac{\pi}{4} + 2n\pi)$

   (b) $(-\frac{3\sqrt{3}}{2},\frac{3}{2}) = (3, \frac{5\pi}{6} + 2n\pi)$

   (c) $(-2,-2\sqrt{3}) = (4, \frac{4\pi}{3} + 2n\pi)$

   (d) $(0,-10) = (10, \frac{3\pi}{2} + 2n\pi)$

   (e) $(-3,0) = \left(3, (2n + 1)\pi\right)$

(f) $(-3,4) = (-5\,,\arcsin \frac{-4}{5} + 2n\pi)$ or $\left(-5\,,(\arctan - \frac{4}{3}) + 2n\pi\right)$

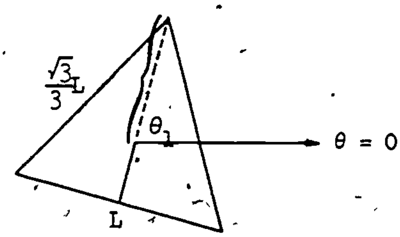
(g) $(-\sqrt{3}\,,1) = (2\,,\frac{5\pi}{6} + 2n\pi)$

(h) $(\sqrt{2}\,,\,-\sqrt{2}) = (2\,,\frac{7\pi}{4} + 2n\pi)$

4. Given the cartesian coordinates (x,y) of a point, formulate unique polar coordinates (r,θ) for $0 \le \theta \le \pi$. (Hint: use $\arccos \frac{x}{r}$.)

$$\left(\frac{y}{|y|}\sqrt{x^2 + y^2}\,,\ \arccos \frac{x}{\sqrt{x^2 + y^2}}\right)$$

5. Determine the polar coordinates of the 3 vertices of an equilateral triangle if a side of the triangle has length L, the centroid of the triangle coincides with the pole, and one angular coordinate of a vertex is θ_1 radians.



The centroid is at the intersection of the medians.

$\frac{2}{3} \cdot L\,\frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{3}L$ = distance of vertex from centroid.

Coordinates of the 3 vertices are:

$(\frac{\sqrt{3}}{3}L\,,\theta_1)$, $(\frac{\sqrt{3}}{3}L\,,\theta_1 + \frac{2\pi}{3})$, $(\frac{\sqrt{3}}{3}L\,,\theta_1 + \frac{4\pi}{3})$.

6. Find equations, in polar coordinates, of the following curves:

(a) $x = c$, c a constant. $\qquad r\cos\theta = c$

(b) $y = c$, c a constant. $\qquad r\sin\theta = c$

(c) $ax + by = c$. $\qquad r(a\cos\theta + b\sin\theta) = c$

(d) $x^2 + (y - k)^2 = k^2$. $\qquad r = 2k\sin\theta$

(e) $y^2 = 4ax$. $\qquad r^2\sin^2\theta = r \cdot 4a\cos\theta$

or $r = \dfrac{4a\cos\theta}{1 - \cos^2\theta}$

(f) $x^2 - y^2 = a^2$. $\qquad r^2\cos 2\theta = a^2$

7. Find equations, in rectangular coordinates, of the following curves:

(a) $r = a.$ $\qquad x^2 + y^2 = a^2$

(b) $r \sin\theta = -5.$ $\qquad y = -5$

(c) $r = 2a \sin\theta.$ $\qquad x^2 + (y - a)^2 = a^2.$ See Number 5(d).

(d) $r = \dfrac{1}{1 - \cos\theta}.$

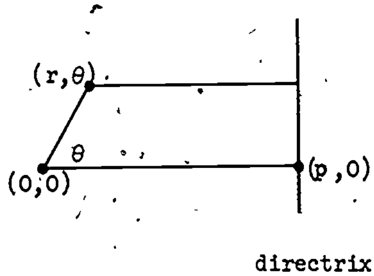
$$\pm\sqrt{x^2 + y^2} - x = 1,$$
$$x^2 + y^2 = x^2 + 2x + 1,$$
$$y^2 = 2x + 1, \quad \text{or}$$
$$x = \frac{1}{2}y^2 - \frac{1}{2}.$$

(e) $r = 2 \tan\theta.$

$$x^2 + y^2 = \frac{4y^2}{x^2} \quad \text{or}$$
$$x^4 + x^2y^2 - 4y^2 = 0.$$

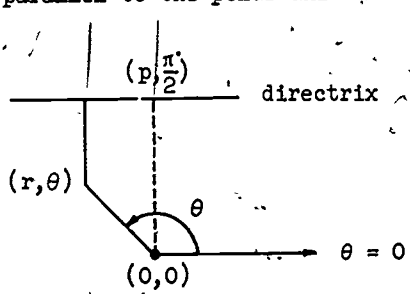
8. Derive an equation in polar coordinates for conic sections with a focus at the pole and directrix perpendicular to the polar axis and p units to the right of the pole.

$$\frac{r}{p - r\cos\theta} = e$$
$$r(1 + e\cos\theta) = ep$$
$$r = \frac{ep}{1 + e\cos\theta}$$



9. Repeat Number 8 if the directrix is parallel to the polar axis and p units above the focus at the pole.

$$\frac{r}{p - r\sin\theta} = e$$
$$r = \frac{ep}{1 + e\sin\theta}$$



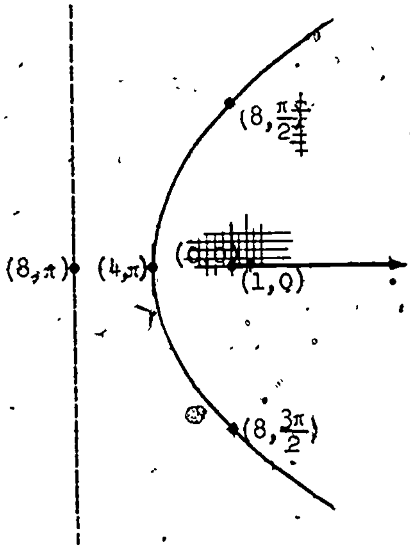
10. Repeat Number 8 if the directrix is parallel to the polar axis and p units below the focus at the pole.

$$r = \frac{ep}{1 - e \sin \theta}$$

11. Discuss and sketch each of the following curves in polar coordinates:

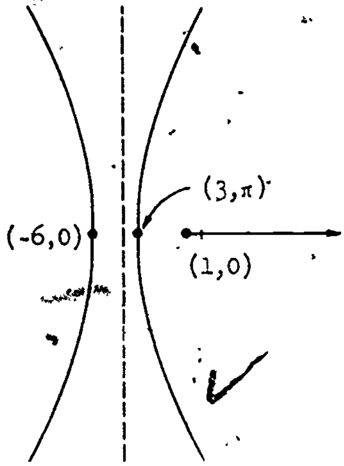
(a) $r = \dfrac{8}{1 - \cos \theta}$

Since $e = 1$, the graph is a parabola and its directrix is 8 units to the left of the pole.



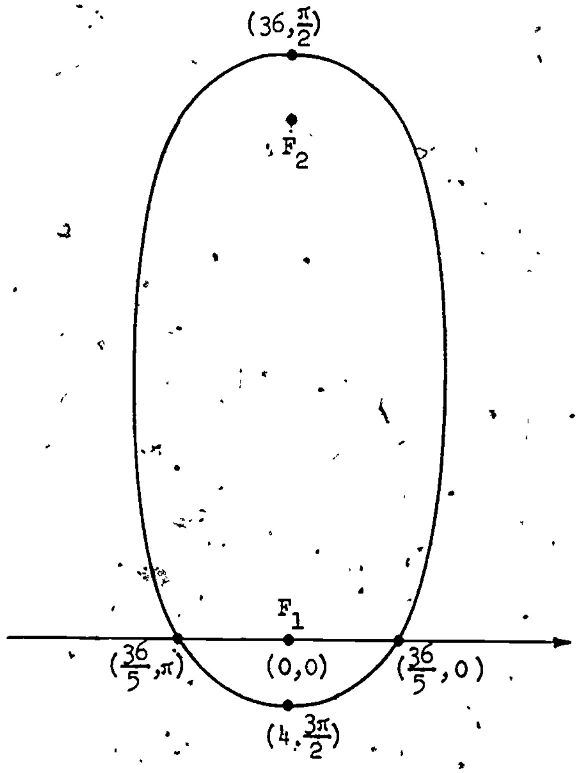
(b) $r = \dfrac{12}{1 - 3 \cos \theta}$

Since $e = 3$, the graph is a hyperbola with its directrix 4 units to the left of the pole.



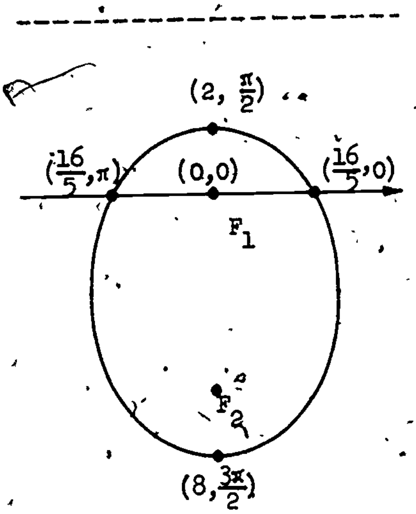
(c) $r = \dfrac{36}{5 - 4\sin\theta} = \dfrac{\frac{36}{5}}{1 - \frac{4}{5}\sin\theta}$

Since $e = \frac{4}{5}$, the graph is an ellipse with its directrix 9 units below the pole.



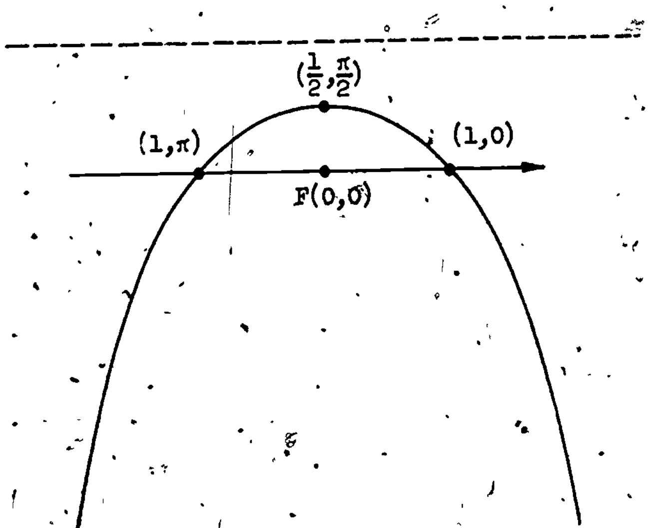
(d) $r = \dfrac{16}{5 + 3 \sin \theta} = \dfrac{\frac{16}{5}}{1 + \frac{3}{5} \sin \theta}$

Since $e = \frac{3}{5}$, the graph is an ellipse with its directrix $\frac{16}{3}$ units above the pole.



(e) $r \sin \theta = 1 - r$ or $r = \dfrac{1}{1 + \sin \theta}$

Since $e = 1$, the graph is a parabola with its directrix one unit above the pole.



12. Certain types of symmetry of curves in polar coordinates are readily detected. For example, a curve is symmetric about the pole if the equation is unchanged when r is replaced by $-r$. What kind of symmetry occurs if an equation is unchanged when

(a) θ is replaced by $-\theta$?

Symmetric about the polar axis.

(b) θ is replaced by $\pi - \theta$?

Symmetric about the line $\theta = \frac{\pi}{2}$.

(c) r and θ are replaced by $-r$ and $-\theta$, respectively?

Symmetric about the line $\theta = \frac{\pi}{2}$.

(d) θ is replaced by $\pi + \theta$?

Symmetric about the pole.

13. Without actually sketching the graphs, describe the symmetries of the graphs of the following equations:

(a) $r^2 = 4 \sin 2\theta$.

Symmetric about the pole. $f(\theta) = f(\theta + \pi)$.
Two lines of symmetry can be demonstrated,
the lines, $\theta = \frac{\pi}{4}$ and $-\frac{\pi}{4}$.

$4 \sin 2(\frac{\pi}{4} - \theta) = 4 \sin 2(\frac{\pi}{4} + \theta)$,

(Earlier, it was shown $\sin(\frac{\pi}{2} + x) = \sin(\frac{\pi}{2} - x)$),

and $4 \sin 2(-\frac{\pi}{4} + \theta) = 4 \sin 2(-\frac{\pi}{4} - \theta)$,

since $\sin(-\frac{\pi}{2} + \theta) = -\sin(\frac{\pi}{2} - \theta)$.

(b) $r(1 - \cos \theta) = 10$

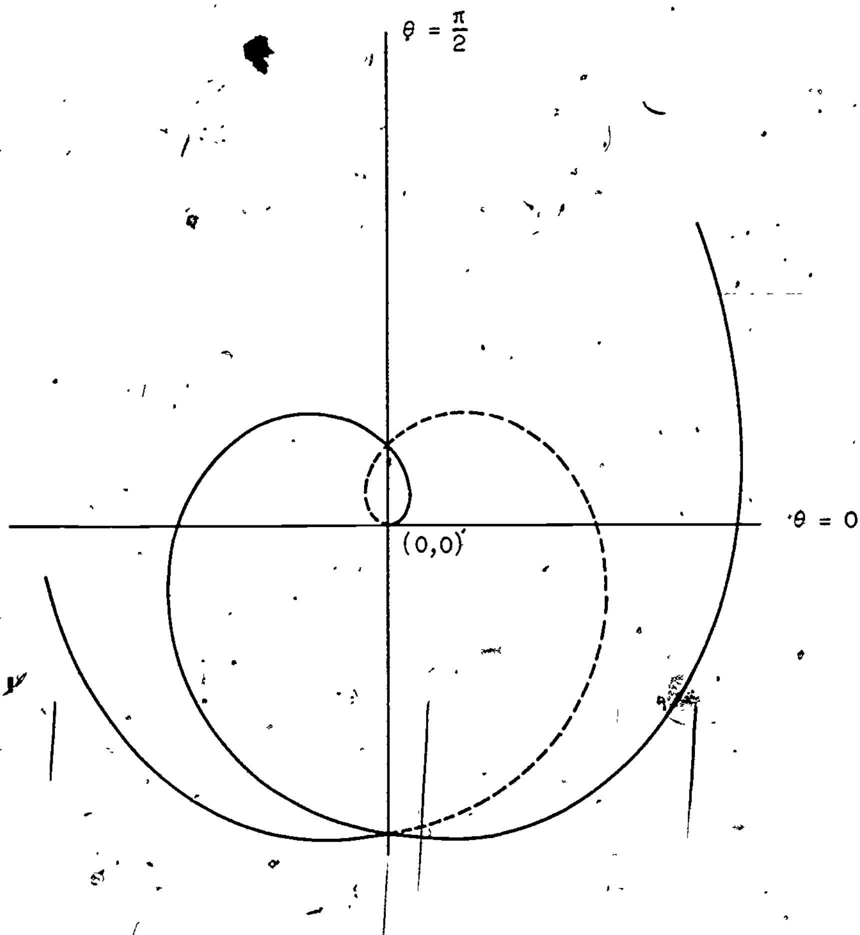
Since $f(\theta) = f(-\theta)$, symmetric about the polar axis.

(c) $r = \cos^2 2\theta$

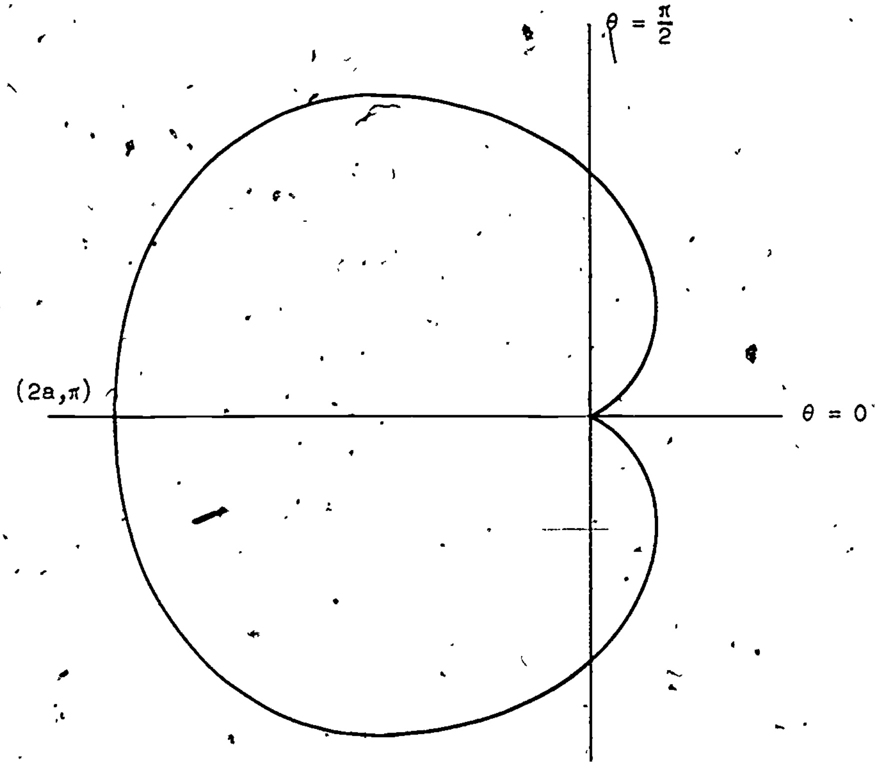
Since $f(\theta) = f(-\theta)$, symmetric about polar axis.
Since $f(\theta) = f(\pi - \theta)$, it is symmetric about the line $\theta = \frac{\pi}{2}$.
Since $\cos(x + \pi) = -\cos x$, and this angle is 2θ, and the cos is squared, the lines $\theta = \frac{\pi}{4}$ and $\theta = \frac{3\pi}{4}$ are axes of symmetry.

14. Sketch the following curves in polar coordinates:

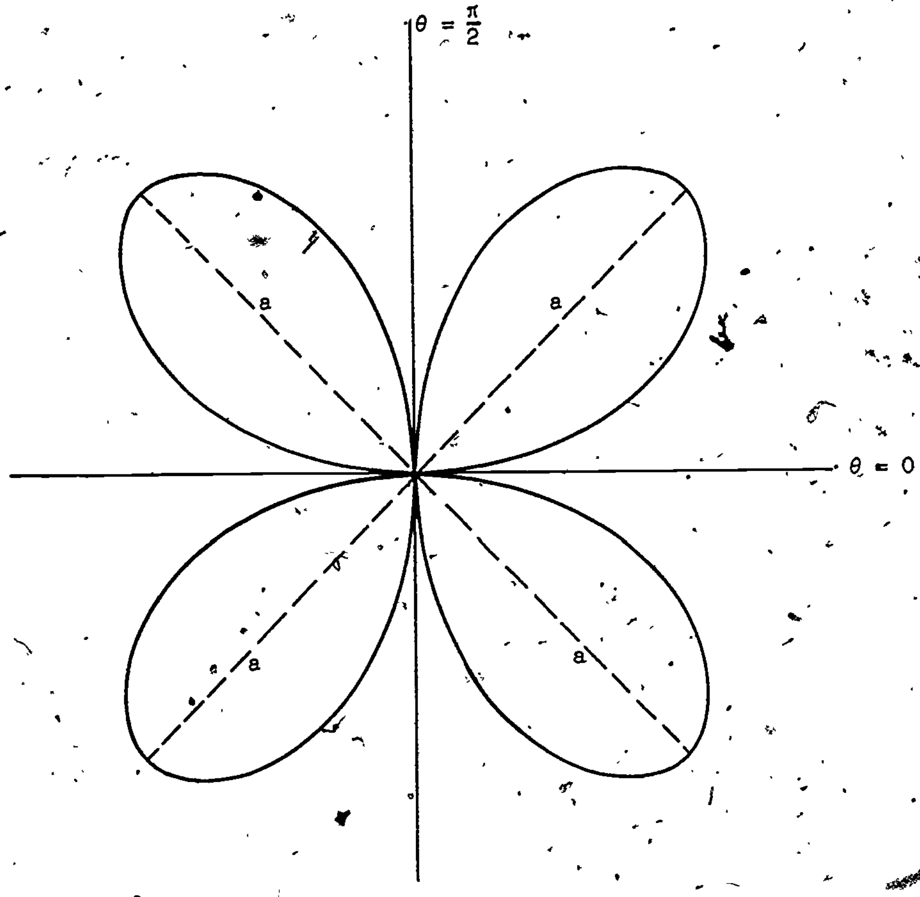
(a) $r = a\theta$



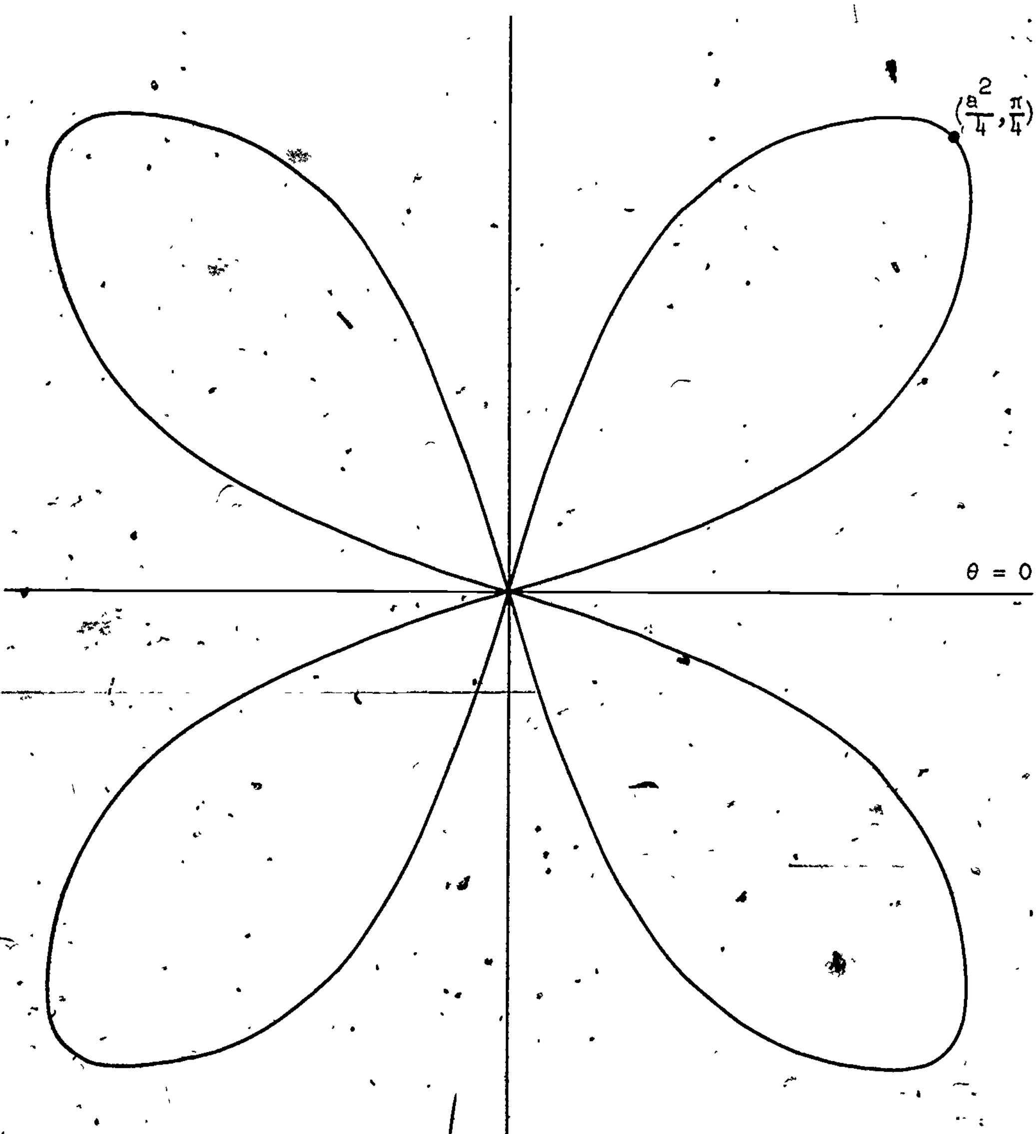
(b) $r = a(1 - \cos\theta)$



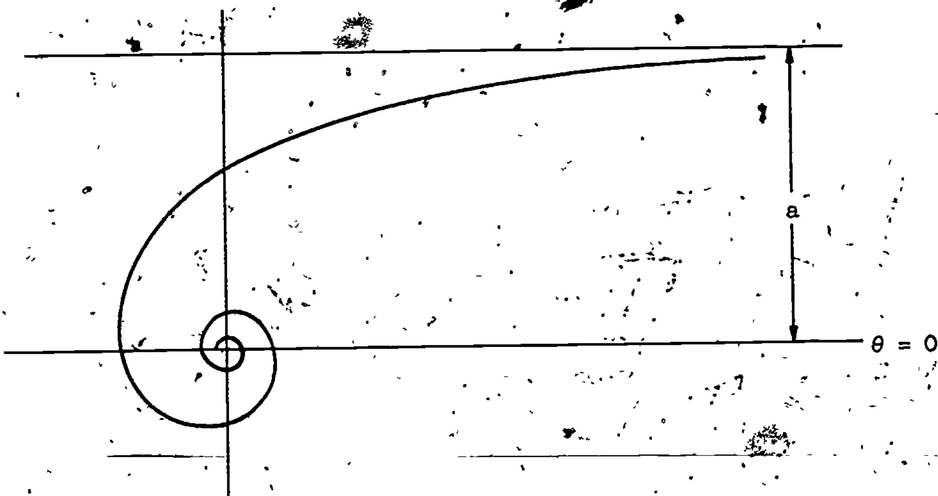
(c) $r = a \sin 2\theta$



(d) $r = a^2 \sin^2 \theta \cos^2 \theta$



(e) $r\theta = a$



15. In each of the following, find all points of intersection of the given pairs of equations. (Recall that the polar representation of a point is not unique.)

(a) $r = 2 - 2\sin\theta$, $r = 2 - 2\cos\theta$.

$\sin\theta = \cos\theta$, $\tan\theta = 1$

$\theta = \frac{\pi}{4}$, $\frac{5\pi}{4}$

Points are $(2 - \sqrt{2}, \frac{\pi}{4})$ and $(2 + \sqrt{2}, \frac{5\pi}{4})$ and since r in each equation may equal 0, the point $(0, \theta)$ is common to both (for different θ).

(b) $r = -2\sin 2\theta$, $r = 2\cos\theta$.

$-2\sin 2\theta = -4\sin\theta\cos\theta = 2\cos\theta$

$\sin\theta = -\frac{1}{2}$ or $\cos\theta = 0$

$\theta = \frac{7\pi}{6}, \frac{11\pi}{6}$ or $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$(0, \frac{\pi}{2}) = (0, \frac{3\pi}{2})$ and $(-\sqrt{3}, \frac{7\pi}{6})$, $(\sqrt{3}, \frac{11\pi}{6})$

(c) $r = 4(1 + \cos\theta)$, $r(1 - \cos\theta) = 3$

$4(1 + \cos\theta)(1 - \cos\theta) = 3$

$1 - \cos^2\theta = \frac{3}{4}$

$\sin^2\theta = \frac{3}{4}$, $\sin\theta = \pm\frac{\sqrt{3}}{2}$

$\theta = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$

$(6, \frac{\pi}{3})$, $(2, \frac{2\pi}{3})$, $(2, \frac{4\pi}{3})$, $(6, \frac{5\pi}{3})$

Teacher's Commentary

Appendix 2

POLYNOMIALS

Solutions Exercises A2-1

1. Eliminate a_0 by subtraction in pairs obtaining

$$+3a_1 - 3a_2 + 9a_3 = +1$$

$$+ a_1 + 3a_2 + 7a_3 = -4$$

$$+2a_1 + 12a_2 + 56a_3 = +2.$$

Eliminate a_1

$$12a_2 + 12a_3 = -13$$

$$6a_2 + 42a_3 = 10.$$

Then

$$-72a_3 = -33$$

$$a_3 = + \frac{33}{72} = + \frac{11}{24}.$$

Since $a_2 + a_3 = - \frac{13}{12}$

$$a_2 = - \frac{13}{12} - \frac{11}{24} = - \frac{37}{24}$$

$$a_1 = -3a_2 - 7a_3 - 4$$

$$= \frac{1}{24} [3.37 - 7.11 - 4.24]$$

$$= - \frac{62}{24} = - \frac{31}{12}.$$

Finally, $a_0 = 3 - a_1 - a_2 - a_3 = \frac{160}{24} = \frac{20}{3}$

2. $\frac{-2}{15}x(x-2)(x-4) + \frac{1}{12}(x+1)x(x-4) + \frac{1}{20}(x+1)x(x-2)$

$$= \frac{x(x-4)}{3}\left[\frac{-2}{5}(x-2) + \frac{1}{4}(x+1)\right] + \frac{1}{20}(x+1)x(x-2)$$

$$= \frac{x(x-4)}{3}\left[\left(\frac{-2}{5}+\frac{1}{4}\right)x + \left(\frac{4}{5}+\frac{1}{4}\right)\right] + \frac{1}{20}(x+1)x(x-2)$$

$$= \frac{x(x-4)}{60}(-3x+21) + \frac{1}{20}(x+1)x(x-2)$$

$$= \frac{x(x-4)}{20}(-x+7) + \frac{1}{20}(x+1)x(x-2)$$

$$= \frac{x}{20}\left[(x-4)(-x+7) + (x+1)(x-2)\right]$$

$$= \frac{x}{20}\left[10x-30\right] = \frac{x}{2}(x-3)$$

Answer: $\frac{x(x-3)}{2}$

Check: $(0,0)$: $\frac{0(-3)}{2} = 0$

$(-1,2)$: $\frac{(-1)(-4)}{2} = 2$

$(2,-1)$: $\frac{2(-1)}{2} = -1$

$(4,2)$: $\frac{4(1)}{2} = 2$

3. $(-1,2)$, $(0,-1)$, $(2,3)$

$$g_1(x) = 2\,\frac{(x-0)(x-2)}{(-1-0)(-1-2)} = \frac{2x(x-2)}{3}$$

$$g_2(x) = -1\,\frac{(x+1)(x-2)}{(0+1)(0-2)} = \frac{1}{2}(x+1)(x-2)$$

$$g_3(x) = 3\,\frac{(x+1)(x-0)}{(2+1)(2-0)} = \frac{1}{2}x(x+1)$$

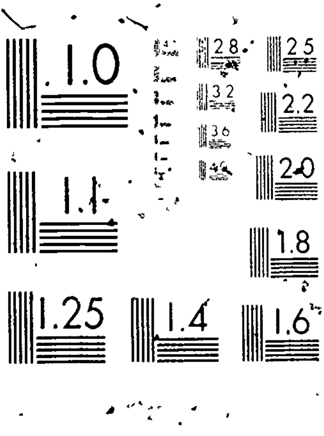
$$g(x) = g_1(x) + g_2(x) + g_3(x)$$

$$= \frac{5x^2-4x-3}{3}$$

Check: $g(-1) = 2$

$g(0) = -1$

$g(2) = 3$



MICROCOPY RESOLUTION TEST CHART

Solutions Exercises A2-2

1. (a) and (b) are unstable; (c), (d), and (e) are stable.

2. (a) -1 of multiplicity two, and 2.
   3rd degree -- sum of multiplicities is 3.

   (b) 1 of multiplicity two, and -2.
   3rd degree -- sum of multiplicities is 3.

   (c) -1 of multiplicity three, and -2.
   4th degree -- sum of multiplicities is 4.

3. (a) 1 of multiplicity two, -2 of multiplicity three.
   (b) 1 of multiplicity three, -2 of multiplicity two.
   The solution set is $\{1,-2\}$ for both equations.

4. (a), (b), (c), (d) are not closed, as shown by the following list of expressions:

   (a) $2x - 1 = 0$ has a root not an integer.

   (b) $x^2 - 2x - 2 = 0$ has roots which are not rational.

   (c) $x^2 + 1 = 0$ has imaginary roots.

   (d) $ix + i = 0$ has a real root.

   (e) is closed. (There is a theorem which establishes this fact.)

5. So far as the specific examples are concerned, $\sqrt[4]{-1}$ and $\sqrt[6]{-1}$ are solutions of $x^4 + 1 = 0$ and $x^6 + 1 = 0$, respectively. In each case, the Fundamental Theorem of Algebra guarantees the existence of a complex zero. If the student is familiar with De Moivre's Theorem, he will know how to obtain, respectively, four and six complex-number solutions. More generally, any root of a complex number is a complex number. It is even the case that all complex powers (or roots) of complex numbers are complex numbers. Hence, "super-complex" numbers are unnecessary. (See Fehr, Howard F., <u>Secondary Mathematics, A Functional Approach for Teachers</u>, D. C. Heath, 1951.)

Solutions Exercises A2-2

1. (a) and (b) are unstable; (c), (d), and (e) are stable.

2. (a) -1 of multiplicity two, and 2.
   3rd degree -- sum of multiplicities is 3.

   (b) 1 of multiplicity two, and -2.
   3rd degree -- sum of multiplicities is 3.

   (c) -1 of multiplicity three, and -2.
   4th degree -- sum of multiplicities is 4.

3. (a) 1 of multiplicity two, -2 of multiplicity three.
   (b) 1 of multiplicity three, -2 of multiplicity two.
   The solution set is $\{1, -2\}$ for both equations.

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   (a) $2x - 1 = 0$ has a root not an integer.

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Solutions Exercises A2-3

1. $[x - (2 + i)][x - (2 - i)][x - 1][x - (3 - 2i)][x - (3 + 2i)]$

$$= (x^2 - 4x + 5)(x - 1)(x^2 - 6x + 13)$$

$$= (x^4 - 10x^3 + 42x^2 - 82x + 65)(x - 1)$$

$$= x^5 - 11x^4 + 52x^3 - 124x^2 + 147x - 65$$

The coefficient of x^4 is -11. The sum of the zeros is $(2 + i) + (2 - i) + 1 + (3 - 2i) + (3 + 2i)$ which is 11. The sum is the negative of the coefficient of x^4. The constant term is -65. The product of the zeros is $(2 + i)(2 - i)(1)(3 - 2i)(3 + 2i)$ which is 65. The product is the negative of the constant term.

2. (a) $x \to x - (2 + 3i) = x - 2 - 3i$

(b) $x \to [x - (2 + 3i)][x - (2 - 3i)] = [(x - 2) - 3i][(x - 2) + 3i]$

$$= x^2 - 4x + 13$$

3. (a) $1, \dfrac{-1 + i\sqrt{3}}{2}, \dfrac{-1 - i\sqrt{3}}{2}$

(b) $-1, \dfrac{1 + i\sqrt{3}}{2}, \dfrac{1 - i\sqrt{3}}{2}$

(c) $2, \dfrac{-1 + i\sqrt{15}}{2}, \dfrac{-1 - i\sqrt{15}}{2}$

(d) $i, -i, 2i, -2i$

(e) 1 of multiplicity two, $3i, -3i$

(f) $-1, i, -i$, each of multiplicity two.

(g) $1, i, -i$, each of multiplicity two. Note that these roots are the negatives of the roots in (f), as would be expected from inspection of the two equations.

4. 8.

5. Using either of the proofs of Theorem A2-4a as a model, two proofs may be shown. Only one is given here.

Given: $f(a + b\sqrt{2}) = 0$, a and b are rational numbers, $b \neq 0$.

To prove: $f(a - b\sqrt{2}) = 0$

Proof: Let $p(x) = [x - (a + b\sqrt{2})][x - (a - b\sqrt{2})]$

$$= [(x - a) - b\sqrt{2}][(x - a) + b\sqrt{2}]$$

$$= (x - a)^2 - 2b^2.$$

The coefficients of $p(x)$ are rational. If $f(x)$ is divided by $p(x)$ we get a quotient $q(x)$ and a remainder $r(x) = hx + k$, possibly of degree 1 (but no greater), where h, k, and all coefficients of $q(x)$ are rational. Thus,

$$f(x) = p(x) \cdot q(x) + hx + k.$$

This is an identity in x. By hypothesis, $f(a + b\sqrt{2}) = 0$, and from $p(x)$ above, $p(a + b\sqrt{2}) = 0$, so we get

$$0 = 0 + ha + hb\sqrt{2} + k.$$

If hb is not zero, we get

$$\sqrt{2} = \frac{-ha - k}{hb}, \text{ where } h, a, k, \text{ and } b$$

are rational, which is impossible. So $hb = 0$, and since $b \neq 0$, h must equal zero, and as a consequence k must equal zero. Therefore,

$$f(x) = p(x) \cdot q(x).$$

Since $p(a - b\sqrt{2}) = 0$, it follows that $f(a - b\sqrt{2}) = 0$.

6. $x \mapsto [x - (3 + 2\sqrt{2})][x - (3 - 2\sqrt{2})] = x^2 - 6x + 1$

7. For $a + b\sqrt{3}$, the proof given in the answer to Exercise 5 will be correct if $\sqrt{3}$ is substituted for $\sqrt{2}$.

For $a + b\sqrt{4}$ there is no comparable theorem since the root is rational and there is no conjugate surd. If a proof like that in Exercise 5 is attempted, it breaks down at the step

$$\sqrt{2} = \frac{-ha - k}{hb}.$$

If 4 is substituted for 2, both sides are rational and the contradiction needed in the proof does not appear.

8. (a) $f : x \to x^2 + (-2 + 2\sqrt{3})x - 3 + 2\sqrt{3}$

(b) $f : x \to x^3 - 5x^2 - 9x - 3$

9. $f : x \to x^4 - 10x^2 + 1$ (The zeros of the function are $\sqrt{3} + \sqrt{2}$, $\sqrt{3} - \sqrt{2}$, and $-\sqrt{3} + \sqrt{2}$, and $-\sqrt{3} - \sqrt{2}$.)

10. (a) (1) Degree 2 $(x \to x^2 - 2\sqrt{2}\,x + 3)$

(2) Degree 2 $(x \to x^2 - 2x + 3)$

(3) Degree 2 $(x \to x^2 - 2\sqrt{2}\,x + 5)$

(b) (1) Degree 4 $(x \to x^4 - 2x^2 + 9)$

(2) Degree 2 $(x \to x^2 - 2x + 3)$

(3) Degree 4 $(x \to x^4 + 2x^2 + 25)$

(Solution for part (1) of (a) and (b)). A polynomial function having only the zero $i + \sqrt{2}$ is $f : x \to x - (i + \sqrt{2})$. This function has imaginary coefficients, but we can obtain from it a function with real coefficients as follows. Write the equation $x - (i + \sqrt{2}) = 0$ in the form $x - \sqrt{2} = i$. Now square both members and rearrange the terms to obtain the equation

$$x^2 - 2\sqrt{2}\,x + 3 = 0.$$

This is the equation corresponding to the function given as the answer to part (1) of (a). It has real coefficients. If we now write this equation in the form $x^2 + 3 = 2\sqrt{2}\,x$ and again square both members, we obtain the equation $x^4 - 2x^2 + 9 = 0$ with rational coefficients. From this we have the answer to part (1) of (b).

An alternative procedure for the two parts of this question depends upon recognition of the fact that if $i + \sqrt{2}$ is a zero of a polynomial function with real coefficients, then the complex conjugate $-i + \sqrt{2}$ is also a zero of the function. Hence, the function of minimum degree having the zeros $i + \sqrt{2}$ and $-i + \sqrt{2}$ will be the answer to part (1) of (a). To obtain a function of minimum degree with rational coefficients, the additional zeros $i - \sqrt{2}$ and $-i - \sqrt{2}$ must be introduced. Hence, the function will be of 4th degree and will have the zeros $i + \sqrt{2}$, $-i + \sqrt{2}$, $i - \sqrt{2}$, and $-i - \sqrt{2}$. This is the function given as the answer to part (1) of (b).